Vector Spaces - Definition

Definition

Let $V$ be a set of vectors equipped with two operations: vector addition and scalar multiplication. Then $V$ is called a vector space if for all vectors $u,v \in V$, the following 10 axioms are satisfied:

1. Closure under addition: $u + v$ is a vector in $V$
2. Commutative property of addition: $u + v = v + u$
3. Associative property of addition: $(u + v) + w = u + (v + w)$
4. Additive identity (zero element): $u + 0 = 0 + u = u$
5. Additive inverse: $u + (-u) = 0$
6. Closure under scalar multiplication: $cu$ is a vector in $V$
7. Distributive property: $c(u + v) = cu + cv$
8. Distributive property: $(c + d)u = cu + du$
9. Associative properties of scalar multiplication: $c(du) = (cd)u$
10. Scalar identity: $1u = u$
Vector Spaces - Examples

Below are some examples of vector spaces. For each item, confirm that it is indeed a vector space by routinely check the 10 axioms for vector spaces.

Example ($\mathbb{R}^n$)

It’s easy to see that for all positive integer $n$, $\mathbb{R}^n$ with standard vector addition and scalar multiplication is a vector space. In fact, they are the models to form the ten axioms of vector space.

Example ($\mathcal{M}_{m,n}$)

The set of all $m \times n$ matrices with matrix addition and scalar multiplication is a vector space. This is straightforward from the definition of these operations.

We sometimes denote $\mathcal{M}_{m,n}$ to be the set of all $m \times n$ matrices and $\mathcal{M}_n$ to be the set of all square $n \times n$ matrices.
Example ($\mathbb{P}_n$)

For $n \geq 0$, let $\mathbb{P}_n$ be the set of all polynomials of the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

where the coefficients $a_0, a_1, \ldots, a_n$ are real numbers.

Given any two polynomials $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $q(t) = b_0 + b_1 t + \cdots + b_n t^n$, we define their sum to be

$$p + q(t) = p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n$$

and the scalar multiplication by $c$, for any scalar $c$, to be

$$(cp)(t) = cp(t) = ca_0 + (ca_1)t + \cdots + (ca_n)t^n$$

Then $\mathbb{P}_n$ with these two operations is a vector space.
Vector Spaces - Examples

Example \((C(-\infty, \infty))\)

Let \(C(-\infty, \infty)\) be the set of all real-valued continuous functions defined on the entire real line.

Given any two functions \(f(x)\) and \(g(x)\), addition is defined by

\[
(f + g)(x) = f(x) + g(x)
\]

and the scalar multiplication by \(c\), for any scalar \(c\), is defined by

\[
(cf)(x) = c[f(x)]
\]

Then \(C(-\infty, \infty)\) with these two operations is a vector space.

Remark: \(C[a, b]\), the set of all continuous functions defined on a closed interval \([a, b]\) is also a vector space under these two operations.
The set of all integers, \( \mathbb{Z} \), with standard addition and multiplication is *not* a vector space because it fails the following property:

A. Closure under addition
B. Commutative property for addition
C. Existence of additive identity - zero element
D. Closure under scalar multiplication
E. Existence of scalar identity
A polynomial of the form

\[ p(t) = a_0 + a_1 t + a_2 t^2 \]

where \( a_0, a_1, a_2 \) are real numbers and \( a_2 \neq 0 \) is called a *second-degree polynomials*. The set of second-degree polynomials with standard addition and multiplication is *not* a vector space since it fails the following property:

A. Closure under addition
B. Commutative property for addition
C. Existence of additive identity - zero element
D. Existence of additive inverse
E. Existence of scalar identity
Question 3

Recall that $\mathbb{R}^2$ with standard vector addition and scalar multiplication is a vector space. However, if we change the operations, then the same set may no longer be a vector space!

Consider $\mathbb{R}^2$ equipped with the standard vector addition. Define the following nonstandard scalar multiplication:

$$c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 0 \end{bmatrix}$$

It fails the following property:

A. Associative property for addition
B. Closure under scalar multiplication
C. Associative property for scalar multiplication
D. Distributive property
E. Existence of scalar identity
Subspaces

Definition

A *nonempty subset* $W$ of a vector space $V$ is called a *subspace* of $V$ if $W$ itself is a vector space under the operations defined in $V$.

To check that $W$ is a subspace of $V$, we only need to check the following conditions:

Theorem

A subset $W$ of a vector space $V$ is a subspace of $V$ if and only if the following three conditions hold:

- The zero vector of $V$ is in $W$
- For all $u, v \in W$, $u + v \in W$
- For all $u \in W$, $cu \in W$ for any scalar $c$
Some questions for next time

1. Under the standard addition and multiplication, is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

2. Suppose a set $V$ is a vector space under addition and scalar multiplication operations. Let $0$ be the zero element of $V$. Is $\{0\}$ a subspace of $V$ under those same two operations?