**Problem 1.** Let $V$ be the set of all $\mathbb{R}^4$ vectors of the form
\[
\begin{bmatrix}
  a - 2b + 5c \\
  2a + 5b - 8c \\
  -a - 4b + 7c \\
  3a + b + c
\end{bmatrix}
\]
for $a, b, c \in \mathbb{R}$. Explain why $V$ is a vector space and find a basis for $V$.

**Problem 2.** For each part of this problem, you are given a vector space $V$, a set $B$ containing the vectors in $V$, and a coordinate vector $[x]_B$. Please do the following:

(i.) First show that $B$ is indeed a basis for $V$,

(ii.) then find the change-of-coordinates matrix from $B$ to the standard basis of $V$, and

(iii.) find the vector $x$ whose coordinates are given by $[x]_B$.

a. $V = \mathbb{R}^3$; $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$; $[x]_B = \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix}$

b. $V = \mathbb{P}_3$; $B = \{1, 1 - t, 2 - 4t + t^2, 6 - 18t + 9t^2 - t^3\}$; $[x]_B = \begin{bmatrix} 5 \\ -4 \\ 3 \\ 1 \end{bmatrix}$

c. $V = \text{Span}\{v_1, v_2, v_3\} = \text{Span}\left\{ \begin{bmatrix} -6 \\ 4 \\ -9 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ -3 \\ 7 \\ -3 \end{bmatrix}, \begin{bmatrix} -9 \\ 5 \\ -8 \\ 3 \end{bmatrix} \right\}$; $B = \{v_1, v_2, v_3\}$; $x = \begin{bmatrix} 4 \\ 7 \\ -8 \\ 3 \end{bmatrix}$
Problem 4. For each part of this problem, you are given vector space $V$, the standard basis $S$ of $V$ and a non-standard basis $B$ of $V$. Find the following:

(i.) $P_{S \leftarrow B}$, the change-of-coordinates matrix from $B$ to the standard basis, and
(ii.) $P_{B \leftarrow S}$, the change-of-coordinates matrix from the standard basis to $B$.

a. $V = \mathbb{R}^3$; $S = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$; $B = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$

b. $V = \mathbb{P}_2$; $S = \{ 1, t, t^2 \}$; $B = \{ 1 - 3t^2, 2 + t - 5t^2, 1 + 2t \}$

Problem 5.

a. Let $B = \{ b_1, b_2 \}$ and $C = \{ c_1, c_2 \}$ be two bases of a vector space $V$ and let $x = 5b_1 + 3b_2$. Suppose $b_1 = -c_1 + 4c_2$ and $b_2 = 5c_1 - 3c_2$. Find $P_{C \leftarrow B}$, the change-of-coordinates matrix from $B$ to $C$, and the coordinates $[x]_C$.

b. Let $B = \{ \begin{bmatrix} 1 \\ -3 \\ 4 \\ \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \}$ and $C = \{ \begin{bmatrix} -3 \\ 4 \\ -3 \\ \end{bmatrix}, \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 7 \end{bmatrix} \}$ be two bases of $\mathbb{R}^3$. Find $P_{C \leftarrow B}$ and $P_{B \leftarrow C}$

Problem 6. Answer the following statements with True or False. If true, give the approximate location in the textbook where a similar statement appears, or refer to a definition or theorem. If false, give a counter-example to show that the statement is not true for all cases.

a. If $A$ is a matrix then $\text{rank}(A)$ equals to the number of non-zero rows in $A$.

b. In $\mathbb{P}_2$, take $f(t) = 3 + t$ and $g(t) = 3t + 3t^2$. Then $f(t)$ and $g(t)$ are linearly dependent since $g(t) = tf(t)$.

c. The columns of $P_{C \leftarrow B}$ are linearly independent.

d. If $A$ is a $5 \times 6$ matrix with four pivot columns then $\text{Col}(A) = \mathbb{R}^4$.

e. If $A$ is a $6 \times 8$ matrix then the smallest possible dimension for $\text{Nul}(A)$ is 2.

f. If $A$ is a $6 \times 4$ matrix then $\text{nullity}(A) \geq 2$. 

2
Problem 7 - Optional (maximum 2 out of 10 points). For this exercise, we shall use MatLab to discover some conjectures about the determinant of a matrix. You need to print out the MatLab codes and result that you obtain and submit them together with your answer to the previous six exercises.

a. First, create a $5 \times 5$ matrix $A$ with randomized integer entries between -10 and 10 and compute $\det(A)$.

b. Compute the following determinants: $\det(A^T)$, $\det(-A)$, $\det(2A)$, $\det(5A)$, $\det(-3A)$ and compare those values to $\det(A)$

c. Using the results from part (b), give your conjectures about the relation between each of $\det(A^T)$, $\det(-A)$, and $\det(kA)$ (where $k$ is a scalar) to $\det(A)$.

d. Test your conjectures using some other randomized matrix with higher dimensions, as well several different values for the scalar $k$. Comment on whether your findings in part (c) still holds.