Instruction. Put your name, PID, assignment title, section number, and TA’s name on top of your homework paper. Make sure your solutions are clear and legible, and show all your work. Credit may not be given for illegible or unsupported answers. You may use calculator, MatLab, or other computational tools to perform the row reduction algorithm.

This assignment has three pages with seven problems.

Problem 1. Compute the determinant of the following matrix, using two methods: row reduction and cofactor expansion:

\[
A = \begin{bmatrix}
2 & 0 & 3 & 2 \\
5 & 1 & 2 & 4 \\
3 & 0 & 1 & 2 \\
5 & 3 & 2 & 1
\end{bmatrix}.
\]

For row reduction method, you must show all row operations and indicate how the determinant changes under each operation. For cofactor expansion method, you must indicate at each step the row/column, across which you are expanding.

Problem 2.

a. Find the area of the parallelogram with vertices \((-2, 1), (0, 4), (1, 3), \) and \((-1, 0)\).

b. Find the volume of the parallelepiped with one vertex at the origin and the adjacent vertices at \((3, 2, -4), (0, 0, 5), \) and \((0, 2, 1)\)

c. Find the adjoint of the following matrix and use it to find the inverse

\[
\begin{bmatrix}
1 & 1 & 3 \\
2 & 2 & 1 \\
0 & 1 & 1
\end{bmatrix}.
\]

Problem 3.

a. Suppose

\[
\begin{vmatrix}
a & b & c \\
d & e & f
\end{vmatrix} = \pi. \text{ Find } \begin{vmatrix}
a + 3d & b + 3e & c + 3f \\
d + 1 & e + 2 & f + 3
\end{vmatrix} \text{ and } \begin{vmatrix}
6 & 4 & 2 \\
f & e & d
\end{vmatrix}.
\]

b. Let \(A\) and \(B\) be \(3 \times 3\) matrices such that \(\det(A) = 3\) and \(\det(B) = 2\). For each of the following, give the value if you have enough information. Otherwise, indicate “not enough information.”

i. \(\det(A + B)\)  
ii. \(\det(A^TB)\)  
iii. \(\det(A^{-1})\)  
iv. \(\det(4A)\)  
v. \(\text{rank}(B)\)

c. Let \(A = \begin{bmatrix}1 & 0 & 1 \\
1 & 1 & 2 \\
1 & 2 & 1\end{bmatrix}\). Find \(\det(B^5)\).
Problem 4.

a. Let \( A = \begin{bmatrix} 4 - x & -4 & -4 \\ 2 & -2 - x & -4 \\ 3 & -3 & -4 - x \end{bmatrix} \). Find all values of \( x \) such that \( \det(A) = 0 \).

b. Let \( A \) be a square matrix such that \( A^T A = I \). Find \( \det(A) \).

c. Let \( A \) and \( B \) be square matrices of the same dimension such that \( B \) is invertible. Show that \( \det(B^{-1}AB) = \det(A) \).

Problem 5.

a. For each of the following, you are given a vector \( \mathbf{v} \) and a matrix \( A \). Determine whether \( \mathbf{v} \) is an eigenvector of \( A \). If \( \mathbf{v} \) is an eigenvector of \( A \), also find the corresponding eigenvalue.

\((i.) \quad \mathbf{v} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix} \)

\((ii.) \quad \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix} \)

b. For each of the following, you are given a scalar \( \lambda \) and a matrix \( A \). Determine whether \( \lambda \) is an eigenvalue of \( A \). If \( \lambda \) is an eigenvector of \( A \), also find the corresponding eigenvector.

\((i.) \quad \lambda = 4, \quad A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} \)

\((ii.) \quad \lambda = 3, \quad A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \)

c. Find the basis for the eigenspace of \( A = \begin{bmatrix} 8 & -10 & -5 \\ 2 & 17 & 2 \\ -9 & -18 & 4 \end{bmatrix} \) corresponding to the eigenvalue \( \lambda = 13 \).

Problem 6.

a. Let \( A \) be an \( n \times n \) matrix and let \( \lambda \) be an eigenvalue of \( A \) with corresponding eigenvector \( \mathbf{x} \neq 0 \). Show that \( \lambda^2 \) is an eigenvalue of \( A^2 \). What is the eigenvector of \( A^2 \) corresponding to \( \lambda^2 \)?

b. Let \( A \) be an invertible \( n \times n \) matrix and let \( \lambda \) be an eigenvalue of \( A \) with corresponding eigenvector \( \mathbf{x} \neq 0 \). Show that \( \lambda^{-1} \) is an eigenvalue of \( A^{-1} \). What is the eigenvector of \( A^{-1} \) corresponding to \( \lambda^{-1} \)?

c. Let \( A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \) and \( P = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \). Compute \( AP \) and use your result to conclude that \( \mathbf{v_1}, \mathbf{v_2}, \) and \( \mathbf{v_3} \) are all eigenvectors of \( A \). Find their corresponding eigenvalues.
Problem 7. Answer the following statements with True or False. If true, give the approximate location in the textbook where a similar statement appears, or refer to a definition or theorem. If false, give a counter-example to show that the statement is not true for all cases.

a. If $A$ is a square matrix and $A^2 = 0$ then $\det(A) = 0$.

b. If the columns of $A$ are linearly dependent then $\det(A) = 0$.

c. $\det(A^{-1}) = -\det(A)$.

d. Let $A$ and $B$ are square matrices of the same dimension. If $AB$ is invertible then both $A$ and $B$ must be invertible.

e. If $Ax = \lambda x$ for some scalar $\lambda$ then $x$ is an eigenvalue of $A$.

f. If $A$ and $B$ are square matrices of the same dimension then $\det(BA) = \det(AB)$

g. If $A$ is a square matrix with $\det(A) = 0$ then $A^2$ is the zero matrix.