Problem 1. For each of the following matrix, please do the following:

i. Find the characteristic polynomial of the given matrix,

ii. Find the eigenvalues together with their multiplicity and corresponding eigenvectors,

iii. Decide whether the given matrix $M$ is diagonalizable. If so, diagonalize it and compute $M^3$ by hand. Do not use MatLab.

(a.) $\begin{bmatrix} -2 & 4 \\ 2 & 5 \end{bmatrix}$; (b.) $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$; (c.) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 4 \\ 0 & 3 & 3 \end{bmatrix}$; (d.) $\begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$; (e.) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Problem 2. Let $v = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

a. Compute $\|v\|^2$, $\|w\|^2$, and $\|v + w\|^2$.

b. Compute $\|-\frac{1}{3}(v - w)\|$ and $\|\frac{1}{3}(v - w)\|$ and compare the result to $\text{dist}(v, w) = \|v - w\|$.

c. Is $\|v + w\| = \|v\| + \|w\|$? If not, then which quantity is larger?

d. Is $\|v - w\| = \|v\| - \|w\|$? If not, then which quantity is larger?

e. Compute $v \cdot w$ and compare $|v \cdot w|$ to $\|v\|\|w\|$.

f. Find the angle $\theta$ between two given vectors, for $0 \leq \theta \leq \pi$.

Problem 3. Determine which of the following matrices are orthogonal, i.e. having orthonormal columns. Then find the inverse of the matrices and briefly comment on the feature of the inverse of an orthogonal matrix that you observe.

$$A = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}; \quad B = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ -2 & 2 & -1 \\ 2 & 1 & -2 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Problem 4. Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. 

a. Show that $B = \{v_1, v_2, v_3\}$ is an orthogonal set in $\mathbb{R}^3$.

b. Find the distance between $v_2$ and $v_3$.

**Problem 5.** Let $A$ be an $8 \times 8$ matrix with characteristic polynomial

$$p(\lambda) = \lambda^2(\lambda - 1)^3(\lambda + 2)^3$$

We know that two of the eigenspaces are each two dimensional. Please answer the following questions and give a reason for your answer.

a. Is it possible that $A$ is diagonalizable?

b. Is it possible that $A$ is singular?

c. For what value of $c$ is it possible to find a nonzero vector $v \in \mathbb{R}^8$ such that $Av = cv$?

**Problem 6.** Given a $3 \times 4$ matrix $A$. During a quiz, students were told to compute a basis for the row space and null space of $A$. This following answer was turned in:

- Basis for $\text{Row}(A)$: $[1 \ 0 \ 0 \ 0]$; $[0 \ 1 \ -1 \ 0]$; $[0 \ 0 \ 0 \ 1]$
- Basis for $\text{Null}(A) = \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{Bmatrix}$

The grader immediately marked the answer wrong. Explain why it is possible to tell the answer is wrong without even knowing the matrix $A$.

**Problem 7.** Suppose $A$ is an $n \times n$ matrix such that $A^2 = O_n$, where $O_n$ is the $n \times n$ zero matrix. Show that $\lambda = 0$ is the only possible eigenvalue of $A$. You need to explain why $0$ is an eigenvalue of $A$ then show that any non-zero scalar $k$ cannot be an eigenvalue of $A$.

**Problem 8.** Let $A$ be a $3 \times 3$ matrix with eigenvectors $u_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

such that $A u_1 = u_1$, $A u_2 = \frac{2}{3} u_2$ and $A u_3 = \frac{1}{3} u_3$.

Find $\lim_{n \to \infty} A^n \begin{bmatrix} 2 \\ 11 \\ -2 \end{bmatrix}$ and recover the matrix $A$.

**Problem 9.** Let $A$ be a $2 \times 2$ matrix with characteristic polynomial $p(\lambda) = \lambda^2 - \lambda - 6$. Show that $A^2 - A - 6I_2 = O_2$, where $O_2$ is the $2 \times 2$ zero matrix. (Hint: diagonalize $A$)