Main topics:

• Solution set of the homogeneous equation $Ax = 0$
• Nullspace of a matrix
• Parametric vector form of the solution
• Particular solution and solution set of a non-homogeneous equation

Here is an example to show that the null space of a matrix can be spanned by more than one vectors.

Let $A$ be a matrix that is row equivalent to

$$
\begin{bmatrix}
1 & 5 & 2 & -6 & 9 & 0 \\
0 & 0 & 1 & -7 & 4 & 8 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Find the solution to $Ax = 0$ and give your answer in parametric vector form.

Answer: The pivots are highlighted above. Here, $x_1, x_3, x_6$ are basic variables; $x_2, x_4, x_5$ are free. By solving for the basic variables in terms of the free variables, we can easily obtain the general solution, as followed.

$$
\begin{aligned}
x_6 &= 0 \\
x_5 &\text{ is free} \\
x_4 &\text{ is free} \\
x_3 &= 8x_6 - 4x_5 + 7x_4 = -4x_5 + 7x_4 \\
x_2 &\text{ is free} \\
x_1 &= -9x_5 + 6x_4 - 2x_3 - 5x_2 = -x_5 - 8x_4 - 5x_2
\end{aligned}
$$

The solution set in parametric vector form is:

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix}
= \begin{bmatrix}
-5x_2 \\
x_2 \\
7x_4 \\
x_4 \\
x_5 \\
0
\end{bmatrix}
= \begin{bmatrix}
-5 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
-8 \\
0 \\
7 \\
1 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
-1 \\
0 \\
-4 \\
0 \\
1 \\
0
\end{bmatrix}.
$$

Since there are three free variables, the nullspace of $A$ is spanned by three vectors. Namely,

$$
Nul(A) = Span \left\{ \begin{bmatrix}
-5 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-8 \\
0 \\
7 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
-4 \\
0 \\
1 \\
0
\end{bmatrix} \right\}.
$$