Definition

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable then the **directional derivative** of $f$ at $x_0$ along the **unit** vector $u = (u_1, \ldots, u_n)$ is given by

$$(D_u f)(x_0) = \nabla f(x_0) \cdot u$$

$$= \left[ \frac{\partial f}{\partial x_1}(x_0) \right] u_1 + \left[ \frac{\partial f}{\partial x_2}(x_0) \right] u_2 + \cdots + \left[ \frac{\partial f}{\partial x_n}(x_0) \right] u_n$$
Find the directional derivative of $f(x, y)$ at $(0, 4)$ in the direction of $u = (3, 4)$. Here, $\nabla f(x, y) = (y \cos(x), 2e^{xy})$

A. 4
B. 5
C. 20
D. $2e^{12}$
E. $8e^{12}$
Let $f$ be a differentiable function with $\nabla f(x_0) \neq 0$ and $u$ be a unit vector. Which of the following is false about the directional derivative $(D_u f)(x_0)$?

A. $(D_u f)(x_0) = 0$ if $\nabla f(x_0) \perp u$

B. If $u$ is in the same direction of $\nabla f(x_0)$ then $f$ increases the fastest/most rapidly

C. If $u$ is in the opposite direction of $\nabla f(x_0)$ then $f$ increases the slowest/least rapidly

D. The maximum value for $(D_u f)(x_0)$ is $\| \nabla f(x_0) \|

E. Choose this if all the above are true
Gradient and Level Set

Theorem

Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be differentiable at \((x_0, y_0)\) and let \( \mathbf{u} \) be a unit vector that is tangent to the level curve \( f(x, y) = c \) at \((x_0, y_0)\). Then

\[
(D_u f)(x_0, y_0) = 0.
\]

Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function with \( \nabla f(x_0, y_0) \neq 0 \).
Let \( C \) be a level curve of \( f \) that contains \((x_0, y_0)\).
Then the tangent line of \( C \) at the point \((x_0, y_0)\) is given by

\[
\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0
\]
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $(x_0, y_0)$. The equation of the tangent plane of $f$ at $(x_0, y_0, f(x_0, y_0))$ is given by

A. $f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0)$

B. $f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0) = 0$

C. $z = f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0)$
Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be a function with \( \nabla f(x_0, y_0, z_0) \neq 0 \). Let \( S \) be a level surface of \( f \) that contains \((x_0, y_0, z_0)\).

Then the gradient \( \nabla f(x_0, y_0, z_0) \) is normal to \( S \) and the tangent plane of \( S \) at the point \((x_0, y_0, z_0)\) is given by

\[
\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0
\]
Example

Find the equation of the tangent plane of the sphere

\[ x^2 + y^2 + z^2 = 1 \]

at the point \( \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \)

Example

Find the equation of the tangent plane of the sphere

\[ e^{y^2}xz^4 - x^4z = 0 \]

at the point \((1, 0, 1)\)