Second Order Partial Derivatives

**Definition**

Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) such that \( \partial f / \partial x, \partial f / \partial y \) exist and are continuous.

- \( f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \): taking the partial derivative w.r.t. \( x \) twice
- \( f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) \): taking the partial derivative w.r.t. \( y \) twice
- \( f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) \): taking the \( x \)-partial first then \( y \)-partial
- \( f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) \): taking the \( y \)-partial first then \( x \)-partial

The last two are called **mixed partial derivatives**.

If all second order partial derivatives exist and are continuous, then we say that \( f \) is of **class \( C^2 \)** (**twice continuously differentiable**).
Example 1

Example

Find all the second partial derivatives of \( f(x, y) = x \ln(y) \)

Solution:

\[
\begin{align*}
  f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = 0; \\
  f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = -\frac{x}{y^2} \\
  f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{1}{y};
\end{align*}
\]

\[
\begin{align*}
  f_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{1}{y}.
\end{align*}
\]
Example 2

Example

Find all the second partial derivatives of \( f(x, y) = \sin(x^2 y) \)

Solution:

\[
\begin{align*}
    f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = 2y \cos(x^2 y) - 4x^2 y^2 \sin(x^2 y) \\
    f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = -x^4 \sin(x^2 y) \\
    f_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = 2x \cos(x^2 y) - 2x^3 y \sin(x^2 y) \\
    f_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = 2x \cos(x^2 y) - 2x^3 y \sin(x^2 y)
\end{align*}
\]
Let $f(x, y) = y^3x - x^2y^2 + 4x^2y$. Then $f_{yy}$ is

A. $y^3 - 2xy^2 + 8xy$
B. $3xy^2 - 2x^2y + 4x^2$
C. $2y^2 + 8y$
D. $3y^2 - 4xy + 8x$
E. $6xy - 2x^2$
Let \( f(x, y) = y^3x - x^2y^2 + 4x^2y \). Then \( f_{yx} \) is

A. \( y^3 - 2xy^2 + 8xy \)
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C. $2y^2 + 8y$
D. $3y^2 - 4xy + 8x$
E. $6xy - 2x^2$
Theorem

If \( f \) is of class \( C^2 \) (twice continuously differentiable) then

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}
\]

This is known as Clairaut’s theorem. In the homework, we shall extend this result to functions in \( C^3, C^4, \ldots \) as well as consider an example where the function fails to satisfy the hypothesis of this theorem.

Explain why there is no function \( f(x, y) \) such that

\[
\frac{\partial f}{\partial x}(x, y) = 2x + 3y \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = 4x + 6y?
\]
If $f$ is of class $C^2$ is called a harmonic function if it satisfies the following Laplace’s equation

$$f_{xx} + f_{yy} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$ 

Is $f(x, y) = x^3 - 3xy^2$ a harmonic function?