Given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Which of the following interpretation for $\frac{\partial f}{\partial x}$ is false?

A. $\frac{\partial f}{\partial x}(x, y)$ is the derivative of $f$ when $y$ is held fixed

B. $\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$

C. $\frac{\partial f}{\partial x}(x, y)$ gives us the **instantaneous** rate of change of $f$ as $x$ changes and $y$ is fixed

D. $\frac{\partial f}{\partial x}$ gives the slope of the tangent line to the graph of $f$ when we move in the direction **parallel to the x-axis**

E. Choose this if all the above are true
Question 2

Suppose

\[ f(x, y) = x^2 y + y^3. \]

Which of the following gives the correct partial derivatives of \( f \)?

A. \( \frac{\partial f}{\partial x}(x, y) = x^2 + 3y^2 \) and \( \frac{\partial f}{\partial y}(x, y) = 2xy \)

B. \( \frac{\partial f}{\partial x}(x, y) = 2xy \) and \( \frac{\partial f}{\partial y}(x, y) = x^2 + 3y^2 \)

C. \( \frac{\partial f}{\partial x}(x, y) = 2xy + y^3 \) and \( \frac{\partial f}{\partial y}(x, y) = x^2 + 3y^2 \)

D. \( \frac{\partial f}{\partial x}(x, y) = 2x + y \) and \( \frac{\partial f}{\partial y}(x, y) = 2xy + x^2 + 3y^2 \)

E. None of the above
Let \( f : \mathbb{R}^2 \to \mathbb{R} \). The equation of the tangent plane to the graph of \( f \) at the point \((x_0, y_0)\) is given by

\[
z = f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0)
\]

Now let \( f(x, y) = x^2 - 2xy + y^2 \). Find the equation to the tangent plane to the graph of \( f \) at the point \((1, 2)\).

A. \( 1 - 2(x - 1) + 2(y - 2) = 0 \)
B. \( 1 - 2(x - 1) + 2(y - 2) = z \)
C. \( -2x + 2y - z - 1 = 0 \)
D. (B) and (C)
E. (A), (B), and (C)
Differentiability for Function of Two Variables

Definition

Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \). We say that \( f \) is **differentiable** at \((x_0, y_0)\) if \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) exist at \((x_0, y_0)\) and

\[
f(x, y) - \left( f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0) \right) \rightarrow 0 \quad \text{as} \quad (x, y) \rightarrow (x_0, y_0).\]

Here, \( L(x, y) := f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0) \) is the **linear approximation** to \( f \).
Question 4

Estimate

\[(0.99)^2 - 2(0.99)(2.01) + (2.01)^2\]

using linear approximation.

Idea: use the tangent plane at \((1, 2)\) from earlier. Here,

\[f(x, y) \approx f(1, 2) + \left[ \frac{\partial f}{\partial x}(1, 2) \right] (x - 1) + \left[ \frac{\partial f}{\partial y}(1, 2) \right] (y - 2)\]

for \((x, y)\) near \((1, 2)\).

A. 0.96
B. 1
C. 1.04
D. 1.0404
Some basic theorems

Theorem

Let \( f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R} \). If \( f \) is differentiable at \((x_0, y_0) \in U\) then \( f \) is continuous at \((x_0, y_0)\).

Theorem

Let \( f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R} \). Suppose \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) exist and are continuous “near” \( x_0, y_0 \) then \( f \) is differentiable at \((x_0, y_0)\).
A Remark

The first theorem from the previous slide tells us that

\[ \text{differentiable} \implies \text{continuous} \]

but not the other way around.

To show that \( f \) is differentiable at \((x_0, y_0)\), we need to check that the partials exist and are continuous in a neighborhood near \((x_0, y_0)\).

Consider the following function

\[
f(x, y) = \begin{cases} 
\frac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0)
\end{cases}
\]

Here, \( f \) is not differentiable at \((0, 0)\). The partials \( \partial f / \partial x \) and \( \partial f / \partial y \) exist but they are not continuous at \((0, 0)\). However, we can still check that \( f \) is continuous at \((0, 0)\).
Gradient & Matrix of Partial Derivatives

Definition

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$. The gradient of $f$ (denote: $\nabla f$) is given by the $1 \times n$ matrix

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$
Gradient & Matrix of Partial Derivatives

Definition

If \( f : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} = (x_1, \ldots, x_n) \mapsto (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \) then matrix of partial derivatives of \( f \) is given by the \( n \times m \) matrix

\[
\begin{bmatrix}
\nabla f_1 \\
\n\nabla f_2 \\
\vdots \\
\n\nabla f_m
\end{bmatrix}
= 
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}
\]

Here, \( Df(\mathbf{x}) \) is called the total derivative or the differential of \( f \).
Example

Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$f(x, y, z) = (ze^x, -ye^z).$$

Compute the matrix of partial derivatives of $f$

Answer:

$$Df(x) = \begin{bmatrix}
    ze^x & 0 & e^x \\
    0 & -e^z & -ye^z 
\end{bmatrix}$$