Given a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. Which of the following interpretation for $\frac{\partial f}{\partial x}$ is false?

A. $\frac{\partial f}{\partial x}(x, y)$ is the derivative of $f$ when $y$ is held fixed

B. $\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$

C. $\frac{\partial f}{\partial x}(x, y)$ gives us the instantaneous rate of change of $f$ as $x$ changes and $y$ is fixed

D. $\frac{\partial f}{\partial x}$ gives the slope of the tangent line to the graph of $f$ when we move in the direction parallel to the $x$-axis

E. Choose this if all the above are true
Question 2

Suppose

\[ f(x, y) = x^2y + y^3. \]

Which of the following gives the correct partial derivatives of \( f \)?

A. \[ \frac{\partial f}{\partial x}(x, y) = x^2 + 3y^2 \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = 2xy \]

B. \[ \frac{\partial f}{\partial x}(x, y) = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x^2 + 3y^2 \]

C. \[ \frac{\partial f}{\partial x}(x, y) = 2xy + y^3 \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x^2 + 3y^2 \]

D. \[ \frac{\partial f}{\partial x}(x, y) = 2x + y \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = 2xy + x^2 + 3y^2 \]

E. None of the above
Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. The equation of the tangent plane to the graph of $f$ at the point $(x_0, y_0)$ is given by

$$z = f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0)$$

Now let $f(x, y) = x^2 - 2xy + y^2$. Find the equation to the tangent plane to the graph of $f$ at the point $(1, 2)$.

A. $1 - 2(x - 1) + 2(y - 2) = 0$
B. $1 - 2(x - 1) + 2(y - 2) = z$
C. $-2x + 2y - z - 1 = 0$
D. (B) and (C)
E. (A), (B), and (C)
Definition

Let $f : \mathbb{R}^2 \to \mathbb{R}$. We say that $f$ is **differentiable** at $(x_0, y_0)$ if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(x_0, y_0)$ and

$$f(x, y) - \left( f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0) \right) \to 0 \quad \text{as} \quad (x, y) \to (x_0, y_0).$$

Here, $L(x, y) := f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0)$ is the **linear approximation** to $f$. 
Estimate

\[(0.99)^2 - 2(0.99)(2.01) + (2.01)^2\]

using linear approximation.

Idea: use the tangent plane at \((1, 2)\) from earlier. Here,

\[
f(x, y) \approx f(1, 2) + \left[ \frac{\partial f}{\partial x}(1, 2) \right] (x - 1) + \left[ \frac{\partial f}{\partial y}(1, 2) \right] (y - 2)
\]

for \((x, y)\) near \((1, 2)\).

A. 0.96
B. 1
C. 1.04
D. 1.0404
Some basic theorems

Theorem

Let $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$. If $f$ is differentiable at $(x_0, y_0) \in U$ then $f$ is continuous at $(x_0, y_0)$.

Theorem

Let $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$. Suppose $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous "near" $x_0, y_0$ then $f$ is differentiable at $(x_0, y_0)$. 
The first theorem from the previous slide tells us that

\[ \text{differentiable} \implies \text{continuous} \]

but not the other way around.

To show that \( f \) is differentiable at \( (x_0, y_0) \), we need to check that the partials exist and are continuous in a neighborhood near \( (x_0, y_0) \).

Consider the following function

\[
f(x, y) = \begin{cases} 
\frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0)
\end{cases}
\]

Here, \( f \) is not differentiable at \((0, 0)\). The partials \( \partial f / \partial x \) and \( \partial f / \partial y \) exist but they are not continuous at \((0, 0)\). However, we can still check that \( f \) is continuous at \((0, 0)\).
Gradient & Matrix of Partial Derivatives

Definition

Let \( f : U \subset \mathbb{R}^n \rightarrow \mathbb{R} \). The \textbf{gradient} of \( f \) (denote: \( \nabla f \)) is given by the \( 1 \times n \) matrix

\[
\nabla f = \begin{bmatrix}
\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n}
\end{bmatrix}
\]
Gradient & Matrix of Partial Derivatives

**Definition**

If $f : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} = (x_1, \ldots, x_n) \mapsto (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ then **matrix of partial derivatives** of $f$ is given by the $m \times n$ matrix

$$Df(\mathbf{x}) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Here, $Df(\mathbf{x})$ is called the **total derivative** or the **differential** of $f$. 
Example

Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$f(x, y, z) = (ze^x, -ye^z).$$

Compute the matrix of partial derivatives of $f$

Answer:

$$Df(x) = \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}$$