Problem 6.b.

\[ \int \sqrt{x^2 + a^2} \, dx = \frac{x\sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \ln \left( x + \sqrt{x^2 + a^2} \right). \]


Problem 7. Let us first consider the following definitions:

1. A differentiable curve \( c(t) \) is called regular if its derivative never vanishes, that is, \( c'(t) \neq 0 \) for any \( t \). If the curve is regular, then the vector \( T(t) = \frac{c'(t)}{\|c'(t)\|} \) is the unit tangent vector to the curve.

2. If \( \|c'(t)\| = 1 \) for all \( t \) then the curve is said to be parametrized by arc length. Whenever the curve is parametrized by arc length, we shall denote this parameter by \( s \). The scalar \( \kappa = \left\| \frac{dT}{ds} \right\| \) is called the curvature of the curve.

3. If \( \kappa \neq 0 \) then the unit vector \( N(t) = \frac{dT/dt}{\|dT/dt\|} \) is called the principal normal vector to the curve.

4. Lastly, we define \( B(t) = T(t) \times N(t) \) to be the unit binormal vector of \( c \).

If a curve \( c(t) \) is regular but not parametrized by arc length, we can introduce a new independent variable so that the new curve is parametrized by arc length. Take a value \( a \) in the domain of the curve and define

\[ s = p(t) = \int_a^t \|c'(u)\| \, du \]

to be the arc length of \( c(t) \) between \( a \) and \( t \).

We have \( \frac{ds}{dt} = \frac{d}{dt} \int_a^t \|c'(u)\| \, du = \|c'(t)\| > 0 \) since the curve is regular. So \( s \) is an increasing function of \( t \) and thus, the inverse \( t = q(s) \) exists.

We now consider the curve \( c_1(s) = c(q(s)) \) which goes through the same points as the original curve but at a different speed. This new speed is given by

\[ \|c_1'(t)\| = \|q'(s)c'(q(s))\| = q'(s)\|c'(q(s))\| = \frac{1}{p'(q(s))}\|c'(q(s))\| = \frac{\|c'(q(s))\|}{\|c'(t)\|} = 1. \]

This shows that the new curve \( c_1(s) \) is the parametrization by arc length of the given curve \( c(t) \).

The curvature \( \kappa = \left\| \frac{dT(t)}{ds} \right\| \) is defined as the rate at which the unit tangent vector changes with respect to arc length. An alternative formula to compute the curvature is

\[ \kappa = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}. \]