CORRECTION OF THE PRACTICE EXAM

1) (a) The limits of the numerator and the denominator both exist:
\[ \lim_{x \to 0} x^2 + 3x = 0 \quad \text{and} \quad \lim_{x \to 0} 2x^2 - 1 = -1. \]
Since the limit of the denominator is different from 0, the limit of the quotient is the quotient of the limits. So
\[ \lim_{x \to 0} \frac{x^2 + 3x}{2x^2 - 1} = \frac{0}{-1} = 0. \]
(b) \( \lim_{x \to \infty} \sin(x) \) does not exist because as \( x \) becomes very large, the function \( \sin \) does not stabilize to a single value; it keeps oscillating between \( -1 \) and \( 1 \).
(c) You can apply the Hopital’s rule:
\[ \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} = \lim_{x \to 9} \frac{\frac{1}{2}x^{-1/2}}{1} = \frac{1}{2} \cdot \frac{1}{\sqrt{9}} = \frac{1}{6}. \]

2) For both questions, first you compute the general formula for the derivative, then you plug in points.
(a) By the chain rule: \( p'(x) = \frac{1}{f(x)} \cdot f'(x) \). So, \( p'(3) = \frac{1}{2}(-5) = -\frac{5}{2} \).
(b) By the quotient rule: \( q'(x) = \frac{f'(x)x - 1.f(x)}{x^2} \). So, \( q'(3) = \frac{(-5)3 - 1.2}{3^2} = -\frac{17}{9} \).

3) (a) \( f'(1) > f'(2) \).
(b) \( \frac{f(1) - f(0)}{1 - 0} \) is greater than \( \frac{f(2) - f(1)}{2 - 1} \).
(c) \( \frac{f(2) - f(1)}{2 - 1} > f'(2) \).

4) (a) By the definition of differentiability, you want to check that the limit \( \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \) exists. Using the third formula for \( f \), you can actually compute the right limit:
\[ \lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - 0}{h} = \lim_{h \to 0^+} \frac{ah}{h} = a. \]
Indeed, when \( h \) is positive, \( f(h) \) is computed using the third formula.
For the left limit, you will use the middle formula. The reason for that is that the left limit should be thought as \( h < 0 \), but approaching 0. So in the end it will be very close to 0, and so, greater than \(-2\):
\[ \lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h) - 0}{h} = \lim_{h \to 0^-} \frac{h^2 + 4h}{h} = \lim_{h \to 0^-} (h + 4) = 0 + 4 = 4. \]
Now all you have to do is to find \( a \) so that the above left and right limits coincide. The answer is then \( a = 4 \).
(b) I leave the redaction of the answer to you. The answer is that \( b = -4 \). My former version of this file indicated that any value of \( b \) works but this is wrong. Sorry about that.

5) (a) Let’s call \( f \) the function that we are studying: \( f(x) = e^{x^3} \). The tangent line must have an equation of the form \( y - y_0 = m(x - x_0) \). Of course the point \( x_0 = 0 \) and \( y_0 = f(x_0) = e^0 = e^0 = 1 \).
For the slope \( m \), first compute the derivative using the chain rule: 
\[ f'(x) = e^{x/3} \cdot \frac{1}{3} \]
Then deduce the slope by plugging in the value of \( x_0 \) for \( x \): 
\[ m = f'(0) = e^{0/3} \cdot \frac{1}{3} = \frac{1}{3} \]
Altogether, the equation of the tangent at 0 is: 
\[ y - 1 = \frac{1}{3}x, \text{ or if you prefer, } \]
\[ y = \frac{1}{3}x + 1. \]

(b) This question requires a good understanding of tangent lines. First note that \( \frac{4}{3} \) is the value of the tangent line at 1, while \( e^{1/3} \) is the value of the function \( f \) itself at 1. So all you have to do is show that the tangent line that you computed in (a) is below the graph of \( f \). For this it is enough to show that \( f \) is concave up between 0 and 1. But a computation gives that:
\[ f''(x) = e^{x/3} \cdot \frac{1}{9} \]
Since the exponential is always positive, we get that \( f'' \) is positive and the function is indeed concave up. So we are done.

From the above exercise you should remember: if a function is concave up on some interval, then all of the tangent lines at points on this interval is below the graph. For concave down it is the contrary.

6) Just apply the product and chain rules when needed to derivate the equation:
\[ -\sin(xy).y + x.\frac{dy}{dx} = 2y.\frac{dy}{dx}. \]
Then solve in \( \frac{dy}{dx} \):
\[ -\sin(xy).y - \sin(xy).x.\frac{dy}{dx} = 2y.\frac{dy}{dx} \]
\[ -\sin(xy).y = \sin(xy).x.\frac{dy}{dx} + 2y.\frac{dy}{dx} = (\sin(xy).x + 2y).\frac{dy}{dx} \]
Therefore,
\[ \frac{-\sin(xy).y}{\sin(xy).x + 2y} = \frac{dy}{dx}. \]

7) To find the critical points, first derivate \( f \): 
\[ f'(x) = 2x. e^{-x} - x^2 e^{-x} = (2x - x^2)e^{-x}. \]
We have to solve \( f'(x) = 0 \). For this we factorize \( f' \) as much as we can: 
\[ f'(x) = x(2 - x)e^{-x}. \]
Now that we have a nice product form, we know that \( f' \) is equal to 0 whenever any of the factors is 0. So the critical points are, either 0 or 2. (Remember that the exponential function never vanishes; it is always positive.)

Now we study each of these points separately to determine if they are local minima of maxima. For this you can make a variation table, it is the best way not to make any mistake. But since I am terrible in drawing tables on a computer, I will do the second derivative test instead. For this we need to calculate the second derivative:
\[ f''(x) = (2 - 2x)e^{-x} + (2x - x^2).(-e^{-x}) = (2 - 4x + x^2)e^{-x}. \]
- At 0 we see that \( f''(0) = 2 \), which is positive. So this is a local minimum;
- At 2, we get \( f''(2) = -2e^{-2} \). Since the exponential is always positive, we get that \( f''(2) < 0 \). Hence 2 is a local maximum.

8) Let us call \( x \) the length of the vertical sides, and \( y \) the horizontal length of the two plots. Then we have
\[ 1200 = 3x + 2y. \]
Moreover the area of the fenced zone is $A = xy$. We want to maximize this area, so we express it as a function of a single variable, by using the relation on $x$ and $y$:

$$A = xy = x \frac{1200 - 3x}{2}.$$ 

Then we maximize this quantity, as a function of $x$. We find that there is only one critical point: $x_0 = 200$.

Finally to check that this is a global maximum of the function, we first try to understand the interval on which we are working: Since we have only 1200 feet of fence, necessarily, $x$ will be less that 400. Also, it has to be positive. So we just have to check that $x_0$ is the maximum of the function $A$ on the closed interval $[0, 400]$. For this we compare $A(0)$, $A(x_0) = A(200)$, and $A(400)$. As $A(0) = A(400) = 0$, the area is maximized at the critical point.