1) (a) Prove that if \( G \) is finite group and \( \lambda(x) \) is a linear character of \( G \), then for any irreducible character \( \chi \) of \( G \), the function \( \chi^\ast \) defined by \( \chi^\ast(\sigma) = \lambda(\sigma)\chi(\sigma) \) for all \( \sigma \in G \) is also an irreducible character of \( G \).

2) The Dihedral group \( D_4 \) of order 8 is generated by two elements \( a \) and \( b \) subject to the relations
\[
(i) \quad a^4 = b^2 = 1 \quad \text{and} \quad (ii) \quad b^{-1}ab = a^3.
\]
Note that (ii) shows that \( ab = ba^3 \) which implies that we can write any element of \( D_8 \) in the form \( b^x a^y \) for some \( x \) and \( y \). Hence, the elements of \( D_8 \) are 1, \( a \), \( a^2 \), \( a^3 \), \( b \), \( ba \), \( ba^2 \), \( ba^3 \).

(a) Show that the conjugacy classes of \( D_8 \) are \( \{1\} \), \( \{a^2\} \), \( \{a, a^3\} \), \( \{b, ba^2\} \), and \( \{ba, ba^3\} \).

(b) The center of \( D_8 \), \( C(D_8) \), is the set of all elements of \( D_8 \) which commute with all the elements of \( D_8 \). Find \( C(D_8) \).

(c) Find the irreducible representations of \( D_8/C(D_8) \) and the lifting of these representations to \( D_8 \).

(d) Construct the character table of \( D_8 \). Describe your methods.

3) Let \( S_2 \times S_2 \) be the Young subgroup of \( S_4 \). For all irreducible representations \( A \) of \( S_2 \times S_2 \), find the character \( \chi_A^{S_4} \) and decompose it into irreducible characters of \( S_4 \).

4) Let \( A : G \to GL_n(\mathbb{C}) \) and \( B : G \to GL_n(\mathbb{C}) \) be two representations of a finite group \( G \) of the same degree. Suppose that for each \( \sigma \in G \), there is a matrix \( D(\sigma) \) such that
\[
D(\sigma)A(\sigma)D(\sigma)^{-1} = B(\sigma).
\]
Prove that \( A \) is similar to \( B \), i.e. there is a fixed invertible matrix \( T \) such that for all \( \sigma \in G \),
\[
TA(\sigma)T^{-1} = B(\sigma).
\]

5) Let \( G \) be a finite group be the finite group of order 12 which is generated by two elements \( a \) and \( b \) defined by the relations
\[
a^6 = 1 \quad a^3 = (ab)^2 = b^2.
\]
(a) Show that \( ba = a^5b \).

(b) Show that the conjugacy classes of \( G \) are \( \{\epsilon\}, \{a^3\}, \{a, a^5\}, \{a^2, a^4\}, \{b, ba^2b, ba^4b\}, \{ab, a^3b, a^5b\} \).

(c) Show that the center \( Z \) of \( G \) is \( Z = \{\epsilon, a^3\} \).

(d) Show that the commutator subgroup \( G' \) of \( G \) is \( G' = \{\epsilon, a^2, a^4\} \) and that \( G/G' \) is isomorphic to the cyclic group \( \mathbb{Z}_4 \).

(e) Find the four linear characters of \( G \) by lifting the characters from \( G/G' \).

(f) Use the orthogonality relations of the characters to complete the character table of \( G \). (Hint: Use both sets of orthogonality relations. You can also use problem 1.)