1) Let $G$ be a finite group. Suppose that $A : G \to M_n(C)$ is a map such that

(i) $A(\epsilon)$ is non-zero and

(ii) For all $\alpha, \beta \in G$, $A(\alpha)A(\beta) = A(\alpha \beta)$.

Show that there is a non-singular matrix $\Theta \in M_n(C)$ such that for all $\alpha \in G$,

$$\Theta A(\alpha) \Theta^{-1} = \begin{bmatrix} B(\alpha) & 0 \\ 0 & 0 \end{bmatrix}$$

where $B$ is a representation of $G$.

(2) Let $C_3$ be the cyclic group of order 3, say $C_3 = \{\epsilon, a, a^2\}$, and $R$ be the real numbers. Show that the representation $A : C_3 \to GL_2(R)$ defined by

$$A(a) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

is irreducible over $R$.

(3) Let $G$ be subgroup of the symmetric group $S_n$. Suppose that $V$ has a basis $v_1, \ldots, v_n$ such that

$$\sigma v_i = v_{\sigma(i)} \quad \forall \sigma \in G.$$ 

Let $u_i = v_i - v_n$ for $i = 1, \ldots, n - 1$ and let $U$ be the subspace generated by $\{u_1, \ldots, u_{n-1}\}$. (a) Show that $U$ is an $n - 1$-dimensional $G$-module.

(b) In the particular case where $G = S_4$, find the $3 \times 3$ matrices that describe the action of

$$\tau = (12), \quad \tau = (123), \quad \tau = (12)(34), \quad \tau = (1234)$$

on $U$ relative to the basis $\{u_1, u_2, u_3\}$ and obtain the character values in each case. Use this to show that $U$ is an irreducible $S_4$-module in this case.

(c) Show that in the special case where $G = A_4$ is the alternating group, $U$ is also an irreducible $A_4$-module.

(d) Consider the special case when $A_5$. Is $U$ an irreducible $A_5$-module?

(4) Show that if $\{X_1, \ldots, X_n\}$ is a group of commuting matrices in $GL_n(C)$, there is a $\Theta \in GL_n(C)$ such that for all $j$, $\Theta^{-1}X_j\Theta$ is a diagonal matrix.