1) Compute the character table of $S_5$.
Explain how you computed your answer.

2) Let $G$ be a finite group and $H$ be a subgroup of $G$ such that $|G|/|H| = 2$. Prove the following
(a) $H$ is normal in $G$.
(b) Every conjugacy class $K$ of $G$ having nonempty intersection with $H$ either is a conjugacy class of $H$ or splits into two conjugacy classes of $H$ of equal size. Furthermore, such a conjugacy class $K$ of $G$ does not split in $H$ if and only if some $k \in K$ commutes with some $g \in G - H$.
(c) Let $A : G \to GL_n(C)$ be an irreducible representation of $G$. Then either $A \uparrow_G^H$ is irreducible or is the sum of two inequivalent irreducible representations of $H$. Furthermore, $A \uparrow_G^H$ is irreducible if and only if $\chi^A(g) \neq 0$ for some $g \in G - H$.

3) (a) Compute the character table of $A_5$.
(b) Explain why the character table of $A_5$ shows that $A_5$ is simple, i.e. that the only normal subgroups of $A_5$ are $\{e\}$ and $A_5$.

4) Let $A^\lambda$ denote the irreducible representation of $S_n$ corresponding to the partition $\lambda$. Decompose $A^{(3,1)} \otimes A^{(2,2)}$ as a sum of irreducible representations of $S_4$.

5) (a) Compute the Frobenius image of $\chi^{A^{(2,3)} \times A^{(1,2)}}\uparrow_{S_6 \times S_3}^{S_8}$.
(b) Explain how the calculation in (a) allow us to decompose $A^{(2,3)} \times A^{(1,2)}\uparrow_{S_6 \times S_3}^{S_8}$ into is irreducible components.

6) (a) Decompose $T \uparrow_{S_2 \times S_3 \times S_3}^{S_8}$ into a sum of irreducible representations of $S_8$.
(b) Decompose $Alt \uparrow_{S_2 \times S_3 \times S_3}^{S_8}$ into a sum of irreducible representations of $S_8$.

7) Prove the following converse of Schur’s lemma. Let $A : G \to GL_n(C)$ be a representation of a finite group $G$ with the property that only scalar multiples of the identity, $cI$, commute with all $A(g)$ for $g \in G$. Show that $A$ is irreducible.

8) Let $D_8$ be the group of rotations and reflections of a necklace with four beads.

Figure 1: Necklace with 4 beads.
(a) Show that $D_8$ has five conjugacy classes.

$C_1 = \{(1)(2)(3)(4)\}$
$C_2 = \{(1,3)(2,4)\}$
$C_3 = \{(1,2,3,4) ; (1,4,3,2)\}$
$C_4 = \{(1)(2,4)(3) ; (1,3)(2)(4)\}$
$C_5 = \{(1,4)(2,3) ; (1,2)(3,4)\}$

(b) Consider the natural representation of $A$ of $D_8$, i.e for any $\sigma \in D_8$, $A(\sigma)$ is the permutation matrix such that $A(\sigma)_{i,j} = \chi(i = \sigma(j))$.

(bi) Find the values of $\chi^A$ on the conjugacy classes $C_i$.

(bii) Show that the trivial representation $T$ occurs with multiplicity 1 in $A$.

(biii) Show that $\chi^A - \chi^T$ is not the character of an irreducible representation of $D_8$

(c)

(ci) Show that $N = C_1 \cup C_2$ is a normal subgroup of $D_8$ and $D_8/N$ is isomorphic to $Z_2 \times Z_2$.

(cii) Give a table of the values of the lifting of the irreducible characters of $Z_2 \times Z_2$ to $D_8$ on the conjugacy classes of $D_8$.

(d) Give a complete table of the irreducible characters of $D_8$.

(9) More generally, let $D_{4n}$ be the group of rotations and reflections on $2n$ beads. Let $\tau = (1, 2, \ldots, 2n)$ be the elements which generates all the rotations and let $f$ be the reflection about bead 1. Thus the cycle structure of $f$ consists of 2 one cycles (1) and $(n+1)$ plus $n-1$ 2-cycles $(n+1-i, n+1+i)$ for $i = 1, \ldots, n-1$.

![Figure 2: Necklace with 2n beads.](image)

a) Show that $\tau f \tau = f$ and hence $D_{4n} = \{\tau^i, f \tau^i : i = 0, \ldots, 2n - 1\}$.

(b) Find the conjugacy classes of $D_{4n}$

(c) Let $\omega$ be a $2n$ root of unity. Then show that the following represent all the irreducible represent-
There are $4n$ 1-dimensional representations.

$$A^1(\tau^k) = 1, \quad A^1(f \tau^k) = 1 \quad k = 1, \ldots, 2n - 1$$

$$A^2(\tau^k) = 1, \quad A^2(f \tau^k) = -1 \quad k = 1, \ldots, 2n - 1$$

$$A^3(\tau^k) = (-1)^k, \quad A^1(f \tau^k) = (-1)^k \quad k = 1, \ldots, 2n - 1$$

$$A^4(\tau^k) = (-1)^k, \quad A^4(f \tau^k) = (-1)^{k+1} \quad k = 1, \ldots, 2n - 1$$

There are $n - 1$ 2-dimensional representations $A^{4+j}, j = 1, \ldots, n - 1$.

$$A^{4+j}(\tau^k) = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-jk} \end{bmatrix}, \quad A^{4+j}(f \tau^k) = \begin{bmatrix} 0 & \omega^{-jk} \\ \omega^k & 0 \end{bmatrix}$$

10) Let $G = \{g_1, \ldots, g_h\}$ be a finite group. Introduce variables $x_{g_1}, \ldots, x_{g_h}$ and consider the $k \times k$ matrix

$$X = [x_{g_i g_j^{-1}}].$$

We can write $X = \sum_{i=1}^k A(g_i)x_{g_i}$. This defines a map $A : G \to GL(\mathbb{C})$ by sending $g_i \to A(g_i)$.

(a) Show that $A$ is the left regular representation of $G$.

(b) Show that

$$\text{det}(X) = \prod_{\nu=1}^h \text{det}(\sum_{g \in G} A^{(\nu)}(x_g)x_g)^{n_\nu}$$

where $A^{(1)}, \ldots, A^{(h)}$ are a complete set of representatives of the irreducible representations of $G$ and $n_\nu = \dim(A^{(\nu)})$ for $\nu = 1, \ldots, h$.

(c) Use part (b) to show that

$$\text{det}\begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \ldots & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \ldots & x_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \ldots & x_0 \end{bmatrix} = \prod_{r=0}^{n-1} (x_0 + \epsilon^r x_1 + \epsilon^{2r} x_2 + \ldots + \epsilon^{(n-1)r} x_{n-1})$$

where $\epsilon = e^{2\pi i/n}$. 

3