1) Find a basis for $I(V(x^5 - 2x^4 + 2x^2 - x, x^5 - x^4 - 2x^3 + 2x^2 + x - 1))$ in $\mathbb{C}[x]$ (Hint: Use the results from the first assignment).

2) Compute by hand the remainder on the division of the given polynomial $f$ by the ordered set $F$, first using $lex$ order and then using $glex$ order.

(a) $f = x^5y^3 + x^3y^2 - y + 1$ and $F = (xy^2 - x, x - y^2)$.

(b) Repeat the problem with the order of $F$ reversed.

3) Let $\mathbb{R}$ denote the real numbers. (a) Show that $X = \{(x, x) : x \neq 0\} \subseteq \mathbb{R}^2$ is not an affine variety. (Hint: for a contradiction show that $X = V(f_1, \ldots, f_s)$, then for all $i$, $f_i$ vanishes on $(0, 0)$ by considering the polynomial $g_i(x) = f_i(x, x)$.

(b) Show that $W = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is not an affine variety.

4) In this problem, you will show that all polynomial parametric curves in $k^2$ are contained in an affine algebraic variety.

(a) Show that the number of distinct monomials $x^e y^f$ of total degree $\leq m$ is equal to $\binom{m+2}{2}$.

(b) Show that if $f(t)$ and $g(t)$ are polynomial of degree $\leq n$, then for $m$ large enough the monomials

$(f(t))^e (g(t))^f$

with $e + f \leq m$ form a linearly dependent set in $k[t]$.

(c) Deduce from part (b) that if $C : x = f(t), y = g(t)$ is any polynomial paramatized curve, then $C$ is contained in $V(F)$ for some $F \in k[x, y]$.

5) Let $I \subseteq k[x_1, \ldots, x_n]$ be a principle ideal. Show that any finite subset of $I$ which contains a generator for $I$ is a Groebner basis for $I$. 


6) Let $f_1, f_2, \ldots$ be an infinite sequence of polynomials in $k[x_1, \ldots, x_n]$ and let $V(f_1, f_2, \ldots) = \{(a_1, \ldots, a_n) \in k^n : f_i(a_1, \ldots, a_n) = 0 \text{ for } i = 0, 1, \ldots\}$. Show that there is some finite $n$ such that $V(f_1, \ldots, f_n) = V(f_1, f_2, \ldots)$.

7) Find reduced Groebner basis for the following ideals in both lex and graded reverse lex orders where $x > y > z$.

(a) $I = \langle x - z^4, y - z^2 \rangle$

(b) $I = \langle xy - z, xy^2 - z^2 \rangle$.

(c) $I = \langle x^2 + yz, x^4 + 2x^2y + y^2, x + 5y + z \rangle$.

(d) $I = \langle x - z^4, y - z^5 \rangle$.

(Do (a) and (b) by hand and (c) and (d) by computer.)