

(1) Prove that there is an absolute constant  $c > 0$  with the following property. Let  $A$  be an  $n \times n$  matrix with pairwise distinct entries. Then there is a permutation of the rows of  $A$  so that no column in the permuted matrix contains an increasing subsequence of length  $c\sqrt{n}$ . (This problem 10 on page 12 of Alon-Spencer).

(2) Let  $X$  be a random variable with expectation  $E(X) = 0$  and variance  $\sigma^2$ . Prove that for all  $\lambda > 0$ ,

$$\Pr(X \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

(3) Let  $v_1 = (x_1, y_1), \dots, (x_n, y_n)$  be elements of  $\mathbb{Z}^2$  so that none of the  $x_j$  or  $y_j$  are greater than  $\frac{2^{n/2}}{100\sqrt{n}}$  in absolute value. Show that there exists two disjoint sets  $I, J \subseteq \{1, \dots, n\}$  such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

(Hint: Consider  $(X, Y) = (\sum_{i=1}^n \epsilon_i x_i, \sum_{i=1}^n \epsilon_i y_i)$  where  $\epsilon_i \in \{0, 1\}$  are chosen independently and uniformly at random.)

The next three problems are applications of Lovàsz Local Lemma.

(4) Prove that if every edge of a  $k$ -uniform hypergraph  $H = (V, E)$  intersects at most  $2^{k-3}$  other edges, then  $H$  is two colorable.

(5) Let  $k \geq 2$  and  $H = (X, E)$  be a hypergraph such that for all  $e \in E$ ,  $|e| \geq k$ . Also assume the for each point  $v$

$$\sum_{e \in E, v \in e} (1 - 1/k)^{-|e|} 2^{-|e|+1} \leq 1/k.$$

Then show that  $H$  is two-colorable.

(6) Let  $N, n, k$  and  $t$  be positive integers where  $N > n \geq k \geq 2$ . Let  $X$  be set of cardinality  $N$  and  $Y$  be a set of cardinality  $n$ . We say that a function  $f : X \rightarrow Y$  separates a subset  $S$  of  $X$  if  $f$  is an injection when restricted to  $S$ , i.e.  $|S| = |f(S)|$ . Otherwise we say that  $f$  reduces  $S$ .

A  $(N, n, k)$ -perfect hash family of size  $t$  is a sequence  $f_1, f_2, \dots, f_t$  of functions from  $X$  to  $Y$  with the property that for all sets  $S \subseteq X$  such that  $|S| = k$ , there is at least one of  $f_1, f_2, \dots, f_t$  separates  $S$ . Show that an  $(N, n, k)$ -perfect hash family of size  $t$  exists when

$$t \geq \frac{2 + \ln\left(\binom{N}{k} - \binom{N-k}{k}\right)}{\ln(n^k) - \ln(n^k - k! \binom{n}{k})}.$$