Permutation Enumeration and Symmetric Functions

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$\sigma = \sigma_1 \ldots \sigma_n \in S_n$

$$\begin{array}{ll}
Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} & Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\} \\
des(\sigma) = |Des(\sigma)| & rise(\sigma) = 1 + |Rise(\sigma)| \\
maj(\sigma) = \sum_{i \in Des(\sigma)} i & comaj(\sigma) = \sum_{i \in Rise(\sigma)} i \\
rlmaj(\sigma) = \sum_{i \in Des(\sigma)} n - i & rlcomaj(\sigma) = \sum_{i \in Rise(\sigma)} n - i \\
inv(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j) & coinv(\sigma) = \sum_{i<j} \chi(\sigma_i < \sigma_j) \\
exc(\sigma) = |\{i : i < \sigma_i\}| & dec(\sigma) = |\{i : i > \sigma_i\}|
\end{array}$$

where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Also if $\alpha^1, \ldots, \alpha^k \in S_n$, then we shall write $comdes(\alpha^1, \ldots, \alpha^k) = |\bigcap_{i=1}^{k} Des(\alpha^i)|$.

$$[n]_q = 1 + q + \cdots + q^{n-1} = \tfrac{1-q^n}{1-q}$$

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \tfrac{[n]_q!}{[k]_q![n-k]_q!}$$

$$\begin{bmatrix} n \\ \lambda_1, \ldots, \lambda_\ell \end{bmatrix}_q = \tfrac{[n]_q!}{[\lambda_1]_q! \cdots [\lambda_\ell]_q!}$$

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + p^1 q^{n-2} + q^{n-1} = \tfrac{p^n - q^n}{p-q}.$$

1) $\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\sigma \in S_n} x^{des(\sigma)} = \frac{1-x}{-x+e^{u(x-1)}}$

(2) (Carlitz 1970)
$\sum_{n=0}^{\infty} \frac{u^n}{(n!)^2} \sum_{(\sigma,\tau)\in S_n \times S_n} x^{comdes(\sigma,\tau)} = \frac{1-x}{-x+J(u(x-1))}$.

3) (Stanley 1979) $\sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} \sum_{\sigma \in S_n} x^{des(\sigma)} q^{inv(\sigma)} = \frac{1-x}{-x+e_q(u(x-1))}$.

4) (Stanley 1979)
$\sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} \sum_{\sigma \in S_n} x^{des(\sigma)} q^{coinv(\sigma)} = \frac{1-x}{-x+E_q(u(x-1))}$.

5) (Fedou and Rawlings 1995)
$\sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} \sum_{(\sigma,\tau)\in S_n \times S_n} x^{comdes(\sigma,\tau)} q^{inv(\sigma)} p^{inv(\tau)} =$
$\frac{1-x}{-x+J_{q,p}(u(x-1))}$.

$J(u) = \sum_{n\ge 0} \frac{u^n}{n!n!}, \qquad e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} q^{\binom{n}{2}},$

$$E_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!}, \text{ and } \qquad J_{q,p}(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} q^{\binom{n}{2}} p^{\binom{n}{2}}.$$

6) Foata-Han (1997)

Let $(x,q)_0 = 1$ and $(x,q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$ for $n > 0$.

$$C_n(z,x,q,y,p) = \sum_{(\sigma,\tau)\in S_n\times S_n} z^{comdes(\sigma^{-1},\tau^{-1})} x^{des(\sigma)} q^{rlmaj(\sigma)} y^{rise(\tau)} p^{rlcomaj(\tau)}.$$

$$\sum_{n\geq 0} t^n \frac{C_n(z,x,q,y,p)}{(x,q)_{n+1}(y,p)_{n+1}} = \sum_{i,j\geq 0} \frac{x^i y^j}{1+\sum_{n\geq 1}(t(z-1))^{n-1} \begin{bmatrix} i+1 \\ n \end{bmatrix}_q \begin{bmatrix} j+n \\ n \end{bmatrix}_p}.$$

7) Remmel-Mendes

$R_n(z,x,q,y,p,Q,P) =$
$\sum_{(\alpha,\beta,\gamma,\delta)\in S_n^4} z^{comdes(\alpha^{-1},\beta^{-1},\gamma,\delta)} x^{des(\alpha)} q^{rlmaj(\alpha)} y^{rise(\beta)} p^{rlcomaj(\beta)} Q^{inv(\gamma)} P^{coinv(\delta)}$
and set

$$F^{i,j}(t,q,p,Q,P) = \sum_{n\geq 0} t^n \frac{q^{\binom{n}{2}} Q^{\binom{n}{2}} \begin{bmatrix} i+1 \\ n \end{bmatrix}_q \begin{bmatrix} j+n \\ n \end{bmatrix}_p}{[n]_Q![n]_P!}.$$

Then we can use the combinatorial mechanism described above with 4-tuples of permutations instead of pairs of permutations to prove that

$$\sum_{n\geq 0} \frac{R_n(z,x,q,y,p,Q,P)t^n}{[n]_Q![n]_P!(x,q)_{n+1}(y,p)_{n+1}} = \sum_{i,j\geq 0} x^i y^j \frac{1-t}{-t+F^{i,j}(t(z-1),q,p,Q,P)}.$$

Elementary & homogeneous symmetric functions

The n^{th} elementary symmetric function e_n is defined by

$$\sum_{n\geq 0} e_n t^n = \prod_i (1 + x_i t).$$

The n^{th} homogeneous symmetric function h_n is defined by

$$\sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}.$$

The n^{th} power symmetric function p_n is defined by

$$p_n = \sum_i x_i^n.$$

If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition, then
$h_\lambda = \prod_{i=1}^k h_{\lambda_i}$, $e_\lambda = \prod_{i=1}^k e_{\lambda_i}$, and $p_\lambda = \prod_{i=1}^k p_{\lambda_i}$,

$$\begin{aligned} 1 &= \left(\prod_i \frac{1}{1-x_it}\right)\left(\prod_i (1-x_it)\right) \\ &= \left(\sum_{n\geq 0} h_nt^n\right)\left(\sum_{n\geq 0} e_n(-t)^n\right) \end{aligned}$$

$$\sum_{n\geq 0}\left(\sum_{i=0}^{n}(-1)^ie_ih_{n-i}\right)t^n = 1$$

$\sum_{i=0}^{n}(-1)^ie_ih_{n-i} = 0$ for $n \geq 1$.

$\sum_{n\geq 0} h_nt^n = \frac{1}{\left(\sum_{n\geq 0} e_n(-t)^n\right)}$.

Brenti(1993) Define a ring homomorphism $\xi : \Lambda \to Q[x]$ by setting

$$\xi(e_k) = \frac{(x-1)^{k-1}}{k!}$$

where e_k is the k-th elementary symmetric function and $\xi(e_0) = 1$.

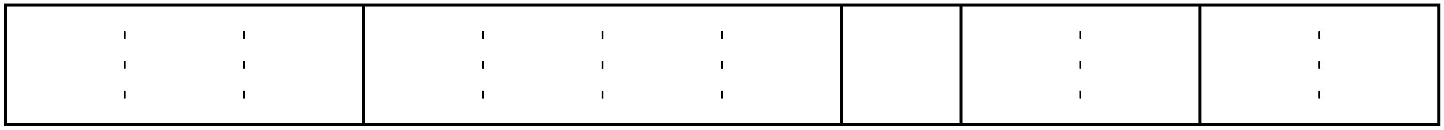
$$n!\xi(h_n) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} \text{ and } \frac{n!}{z_\lambda}\xi(p_\lambda) = \sum_{\sigma \in S_n(\lambda)} x^{\mathrm{exc}(\sigma)} \qquad (1)$$

where if $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ is a partition of n, then $S_n(\lambda)$ is the set of permutations in S_n with cycle type λ, and $z_\lambda = \prod_{i=1}^{n} i^{m_i} m_i!$.

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$.

$\{e_\lambda : \lambda$ a partition$\}$ is a basis for the ring of symmetric functions Λ.

Let $B_{\lambda,n}$ be the number of brick tabloids of shape (n) and type λ:



Let $M_{\lambda,n}$ be the coefficient of e_λ in h_n.

The simple relations displayed earlier can show $M_{\lambda,n}$ satisfies

1. $M_{(n),n} = (-1)^{n-1}$,
2. $M_{\lambda,n} = \sum_{i=1}^{n-1} (-1)^{i-1} M_{\lambda\backslash i,n-i}$.

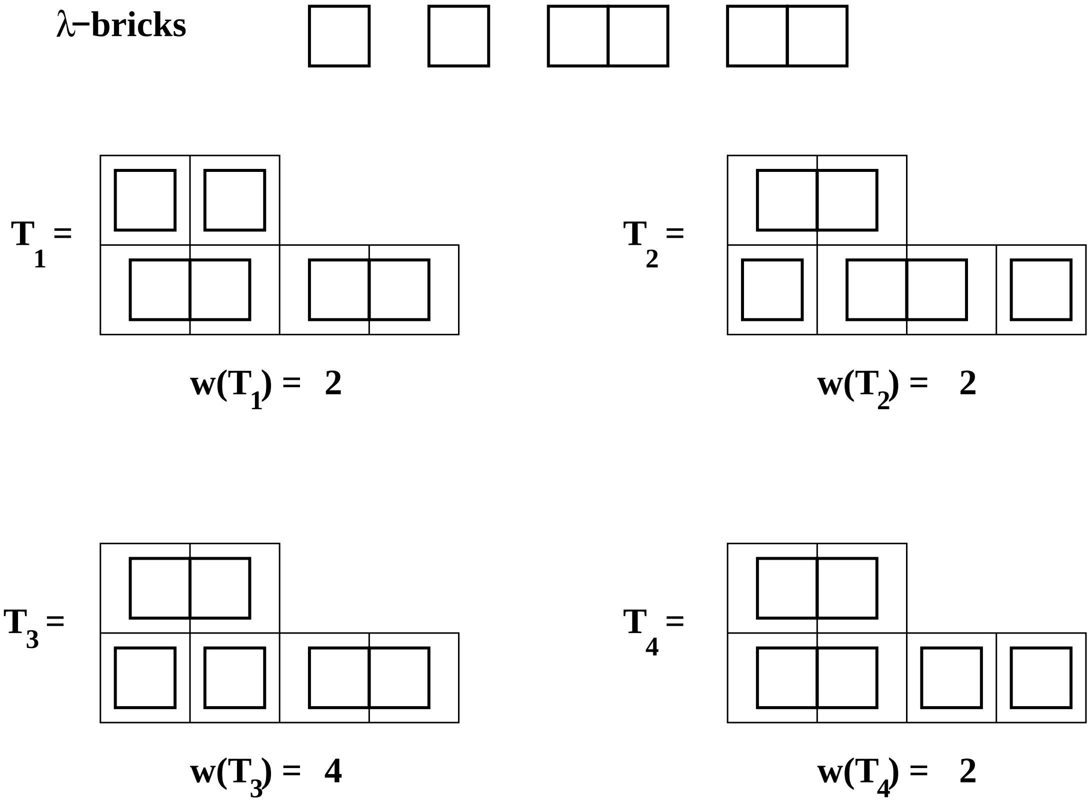
The numbers $(-1)^{n-\ell(\lambda)} B_{\lambda,n}$ satisfy the same recursions.

Therefore,

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda.$$

λ-Brick Tabloids and Weighted λ-Brick Tabloids. Suppose that $\lambda = (1,1,2,2)$ and $\mu = (2,4)$.

$B_{\lambda,\mu} = 4$ and $w(B)_{\lambda,\mu} = 10$



$$\begin{aligned} h_\mu(\overline{x}) &= \sum_{\mu \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,\mu} e_\lambda(\overline{x}) \\ p_\mu(\overline{x}) &= \sum_{\mu \vdash n} (-1)^{n-\ell(\lambda)} w(B)_{\lambda,\mu} e_\lambda(\overline{x}) \end{aligned}$$

(Egecioglu and Remmel 1991)

A link from Λ to permutation enumeration

For $\sigma_1 \cdots \sigma_n \in S_n$,

$des(\sigma)$ is the number of times $\sigma_i > \sigma_{i+1}$,

$ris(\sigma)$ is the number of times $\sigma_i < \sigma_{i+1}$ where $\sigma_{n+1} = n+1$.

Ex. Let $\sigma = 12 \;\; 9 \;\; 7 \;\; 2 \;\; 6 \;\; 8 \;\; 10 \;\; 1 \;\; 3 \;\; 4 \;\; 11 \;\; 5.$

Then $des(\sigma) = 5$ and $ris(\sigma) = 7$.

Let $f_1 : \{0, 1, \ldots\} \to \mathbb{Q}[x, y]$ such that:

$$f_1(n) = \begin{cases} 1 & \text{if } n = 0 \\ y\,(x-y)^{n-1} & \text{if } n \geq 1. \end{cases}$$

Define $\xi^{f_1} : \Lambda \to \mathbb{Q}[x, y]$ as a homomorphism such that

$$\xi^{f_1}(e_n) = \frac{(-1)^{n-1}}{n!} f_1(n).$$

Theorem.

$$n!\xi^{f_1}(h_n) = \sum_{\sigma \in S_n} x^{des(\sigma)} y^{ris(\sigma)}.$$

Proof.

$$\begin{aligned}
n!\xi^{f_1}(h_n) &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \xi^{f_1}(e_\lambda) \\
&= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{\lambda_i - 1}}{\lambda_i!} f_1(\lambda_i) \\
&= \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_1(\lambda_1) \cdots f_1(\lambda_\ell).
\end{aligned}$$

We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

from which we create the following objects:

We have

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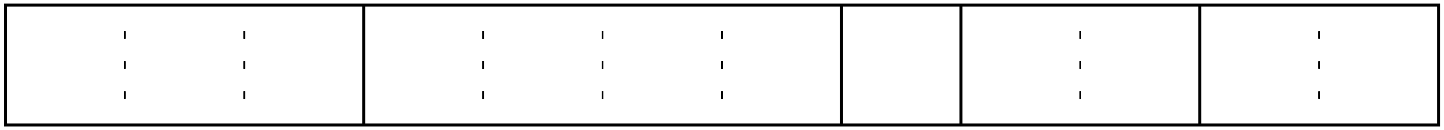

We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

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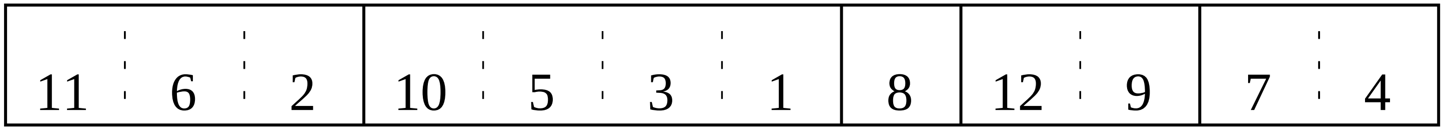


We have

$$n!\xi^{f_1}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$$

from which we create the following objects:

$$\begin{array}{|c:c:c|c:c:c:c|c|c:c|c:c|}
\hline
x & x & y & -y & x & -y & y & y & -y & y & x & y \\
11 & 6 & 2 & 10 & 5 & 3 & 1 & 8 & 12 & 9 & 7 & 4 \\
\hline
\end{array}$$

$f_1(n) = y(x-y)^{n-1}$

Let $\mathcal{T}_{f_1}$ be the set of objects created in this way.

$T = (B, \sigma, L) \in \mathcal{T}_{f_1}$ if

(1) $B = (b_1, \ldots, b_k)$ is a brick tabloid of shape (n).

(2) σ is a permutation which is decreasing within each brick.

(3) L is a labeling such that the last cell of each brick is labeled with a y and every other cell is labeled with x or $-y$.

The weight $w(T)$ of $T \in \mathcal{T}_{f_1}$, $w(T)$, is the product of x,$-y$, and y labels.

An involution $I : \mathcal{T}_{f_1} \to \mathcal{T}_{f_1}$ such that

(a) If $I(T) \neq T$, then $w(T) = -w(I(T))$.

(b) If $I(T) = T$, then $w(T)$ is positive.

I will show that

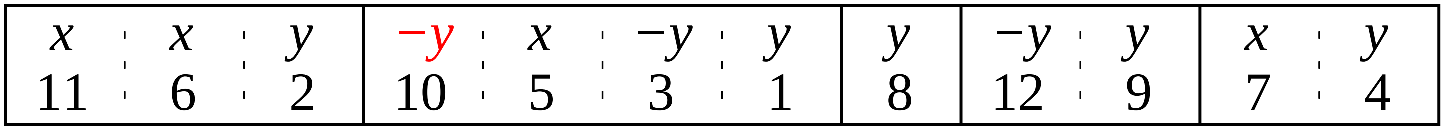
$$\sum_{T \in \mathcal{T}_{f_1}} w(T) = \sum_{T \in \mathcal{T}_{f_1}, I(T)=T} w(T).$$

To define $I(T)$ where $T = (B, \sigma, L)$, scan the cells of T from left to right looking for the first cell c such that either

(i) c is labeled with $-y$ in which case we break the brick b containing c into two bricks b^* and b^{**} where b^* contains all the cells of b up to and including c and b^{**} consists of the remaining cells of b and then we change the label on cell c from $-y$ to y or

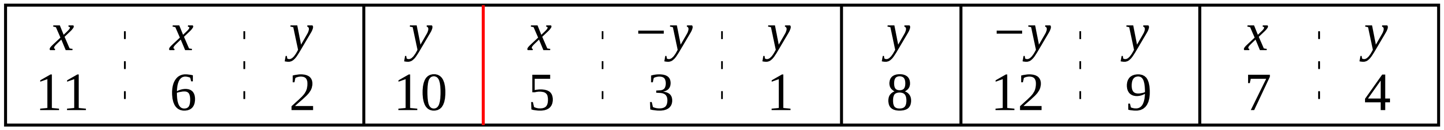
(ii) c is the last cell of a brick b_i which is followed by another a brick b_{i+1} such that σ is decreasing in all the cells corresponding to b_i and b_{i+1} in which case we replace b_i and b_{i+1} by a single brick b and change the label on cell c from y to $-y$.

If neither (i) or (ii) applies, $I(T) = T$.



x 11
x 6
y 2
$-y$ 10
x 5
$-y$ 3
y 1
y 8
$-y$ 12
y 9
x 7
y 4

is sent to



x 11
x 6
y 2
y 10
x 5
$-y$ 3
y 1
y 8
$-y$ 12
y 9
x 7
y 4

Fixed points of I: (a) no $-y$'s and (b) increases between bricks.

x	x	y	x	x	x	y	y	x	y	x	y
11	6	2	10	5	3	1	8	12	7	9	4

A fixed point can be read as an element in S_n:

11 6 2 10 5 3 1 8 12 7 9 4

Therefore,

$$n!\xi^{f_1}(h_n) = \sum_{T \in \mathcal{T}_{f_1}} w(T) = \sum_{\substack{T \text{ is a} \\ \text{fixed point}}} w(T) = \sum_{\sigma \in S_n} x^{des(\sigma)} y^{ris(\sigma)}.$$

$\square$

This gives a generating function:

$$\begin{aligned}
\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in S_n} x^{des(\sigma)}y^{ris(\sigma)} &= \xi^{f_1}\left(\sum_{n\geq 0} h_n t^n\right)\\
&= \xi^{f_1}\left(\sum_{n\geq 0} e_n(-t)^n\right)^{-1}\\
&= \left(1-\sum_{n\geq 1}\frac{t^n}{n!}f_1(n)\right)^{-1}\\
&= \frac{x-y}{x-ye^{t(x-y)}}.
\end{aligned}$$

We just

1. Defined f on $\{0, 1, \dots\}$ and ξ^f on Λ such that

$$\xi^f(e_n) = \frac{(-1)^{n-1}}{n!} f(n)$$

2. Applied ξ^f to $n!h_n$ and decorated brick tabloids

3. Performed an involution to find objects corresponding to permutations

4. Found a generating function from the h_n and e_n relationship

More weighting functions

$\sigma_1 \cdots \sigma_n \in S_n$ is alternating if $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$ for even i.

Ex. 8 6 7 4 5 2 3 1 12 9 11 10 is (even) alternating.

Let $f_2(n) = \begin{cases} 0 & n \text{ odd}, \\ (-1)^{n/2-1} & n \text{ even}. \end{cases}$ and $\xi^{f_2}(e_n) = \dfrac{(-1)^{n-1}}{n!} f_2(n)$.

Theorem. Let A_n be the number of even alternating permutations of n. Then $n!\xi^{f_2}(h_n) = A_n$.

Proof. We have,

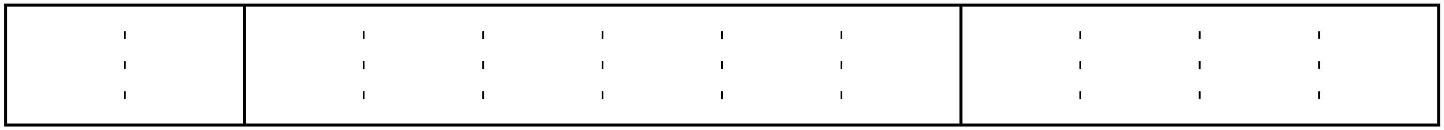
$$n!\xi^{f_2}(h_n) = \sum_{\lambda\vdash n} \binom{n}{\lambda_1,\ldots,\lambda_\ell} B_{\lambda,n} f_2(\lambda_1)\cdots f_2(\lambda_\ell)$$

from which we create the following objects:

$f_2(n) = 0$ if n odd and $f_2(n) = (-1)^{n/2-1}$ if n even.

$$n!\xi^{f_2}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_2(\lambda_1) \cdots f_2(\lambda_\ell)$$

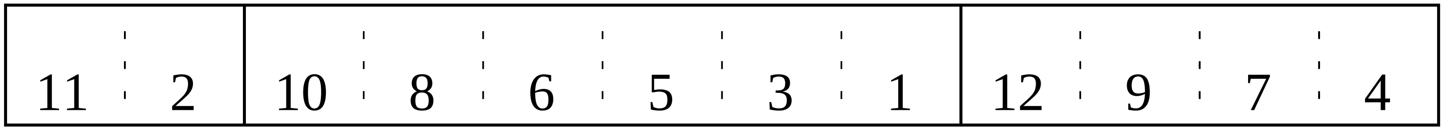
from which we create the following objects:



$f_2 = 0$ if n odd and $f_2 = (-1)^{n/2-1}$ if n even.

$$n!\xi^{f_2}(h_n) = \sum_{\lambda\vdash n} \binom{n}{\lambda_1,\ldots,\lambda_\ell} B_{\lambda,n} f_2(\lambda_1)\cdots f_2(\lambda_\ell)$$

from which we create the following objects:



$f_2 = 0$ if n odd and $f_2 = (-1)^{n/2-1}$ if n even.

$$n!\xi^{f_2}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_2(\lambda_1) \cdots f_2(\lambda_\ell)$$

from which we create the following objects:

1	1	1	−1	1	−1	1	1	1	−1	1	1
11	2	10	8	6	5	3	1	12	9	7	4

$f_2(n) = 0$ if n odd and $f_2(n) = (-1)^{n/2-1}$ if n even.

Perform a similar involution: scan for -1 or a decrease between two bricks:

$$\begin{array}{|cc|cccccc|cccc|}
\hline
1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
11 & 2 & 10 & 8 & 6 & 5 & 3 & 1 & 12 & 9 & 7 & 4 \\
\hline
\end{array}$$

is sent to

$$\begin{array}{|cc|cc|cccc|cccc|}
\hline
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
11 & 2 & 10 & 8 & 6 & 5 & 3 & 1 & 12 & 9 & 7 & 4 \\
\hline
\end{array}$$

A fixed point:

1	1	1	1	1	1	1	1	1	1	1	1
11	2	10	6	8	3	5	1	12	7	9	4

The corresponding permutation:

11 2 10 6 8 3 12 7 9 4

Summing all fixed points gives the desired result. □

The resultant generating function:

$$\begin{aligned}
\sum_{n\geq 0} \xi_{f_2}(h_n) &= \sum_{n\geq 0} \frac{t^{2n}}{(2n)!} A_{2n} \\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^n \xi_{f_2}(e_n)} \\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^{2n}\frac{(-1)^{2n-1}}{(2n)!}(-1)^{n-1}} \\
&= \frac{1}{1+\sum_{n\geq 1}\frac{(-1)^n}{(2n)!}} \\
&= \frac{1}{\cos(t)} \\
&= \sec(t).
\end{aligned}$$

Next we want to show that $(2n+1)!\xi_{f_2}(p_{2n+2}) = A_{2n+1}$.

If $\mu = (\mu_1, \ldots, \mu_k)$, then $2\mu = (2\mu_1, \ldots, 2\mu_k)$.

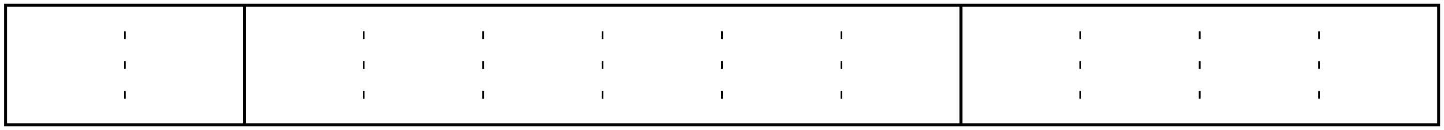
$$(2n{+}1)!\xi_{f_2}(p_{2n+2}) = (2n{+}1)! \sum_{\mu \vdash n+1} (-1)^{2n+2-\ell(\mu)} w(B_{2\mu,(2n)})\xi_{f_2}(e_{2\mu}) =$$

$$(2n{+}1)! \sum_{\mu \vdash n+1} (-1)^{2n+2-\ell(\mu)} \sum_{B=(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,(n+1)}} 2b_k \prod_{i=1}^{\ell(\mu)} \frac{(-1)^{2b_i-1}}{2b_i!}(-1)^{b_i-1} =$$

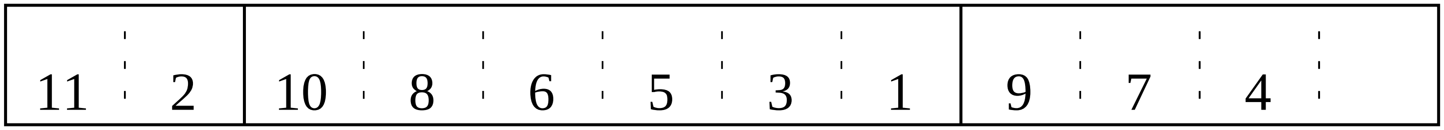
$$\sum_{\mu \vdash n+1} \sum_{B=(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,(n+1)}} \binom{2n+1}{2b_1, \ldots, 2b_{k-1}, 2b_k - 1} \prod_{i=1}^{\ell(\mu)} (-1)^{b_i-1}.$$

$f_2 = 0$ if n odd and $f_2 = (-1)^{n/2-1}$ if n even.

$$\sum_{\mu\vdash n+1}\ \sum_{B=(b_1,\dots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,(n)}} \binom{2n+1}{2b_1,\dots,2b_{k-1},2b_k-1}\prod_{i=1}^{\ell(\mu)}(-1)^{b_i-1}.$$



$$\sum_{\mu \vdash n+1} \sum_{B=(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,(n)}} \binom{2n+1}{2b_1,\ldots,2b_{k-1},2b_k-1} \prod_{i=1}^{\ell(\mu)} (-1)^{b_i-1}.$$



$$\sum_{\mu\vdash n+1}\sum_{B=(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,(n)}}\binom{2n+1}{2b_1,\ldots,2b_{k-1},2b_k-1}\prod_{i=1}^{\ell(\mu)}(-1)^{b_i-1}.$$

1	1	1	−1	1	−1	1	1	1	−1	1	1
11	2	10	8	6	5	3	1	9	7	4	

Perform a similar involution: scan for -1 or a decrease between two bricks:

$$\begin{array}{|cc|cccccc|cccc|}
\hline
1 & 1 & 1 & \color{red}{-1} & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
11 & 2 & 10 & 8 & 6 & 5 & 3 & 1 & 9 & 7 & 4 & \\
\hline
\end{array}$$

is sent to

$$\begin{array}{|cc|cc!{\color{red}\vert}cccc|cccc|}
\hline
1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
11 & 2 & 10 & 8 & 6 & 5 & 3 & 1 & 9 & 7 & 4 & \\
\hline
\end{array}$$

A fixed point:

$$\begin{array}{|cc|cc|cc|cc|cc|cc|}
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
11 & 2 & 10 & 6 & 8 & 3 & 5 & 1 & 7 & 9 & 4 & \\
\hline
\end{array}$$

The corresponding permutation:

$$11 \quad 2 \quad 10 \quad 6 \quad 8 \quad 3 \quad 7 \quad 4 \quad 9$$

Summing all fixed points gives the desired result. ☐

The resultant generating function:

$$\begin{aligned}
\sum_{n\geq 1}\xi_{f_2}(p_n) &= \sum_{n\geq 0}\frac{t^{2n+2}}{(2n+1)!}A_{2n+1}\\
&= \frac{\sum_{n\geq 1}(-1)^{n-1}\nu(n)\xi_{f_2}(e_n)}{1+\sum_{n\geq 1}(-t)^n\xi_{f_2}(e_n)}\\
&= \frac{\sum_{n\geq 1}(-1)^{2n-1}2n\frac{(-1)^{2n-1}}{2n!}(-1)^{n-1}}{1+\sum_{n\geq 1}(-t)^{2n}\frac{(-1)^{2n-1}}{(2n)!}(-1)^{n-1}}\\
&= \frac{\sum_{n\geq 1}\frac{(-1)^{n-1}t^{2n}}{(2n-1)!}}{1+\sum_{n\geq 1}\frac{(-1)^{n}}{(2n)!}}\\
&= \frac{t\sin(t)}{\cos(t)}.
\end{aligned}$$

$$\sum_{n\geq 0} A_{2n+1}\frac{t^{2n+2}}{(2n+1)!} = \frac{t\sin(t)}{\cos(t)}.$$

Dividing both sides by t, we obtain

$$\sum_{n\geq 0} A_{2n+1}\frac{t^{2n+1}}{(2n+1)!} = \frac{\sin(t)}{\cos(t)} = \tan(t).$$

B_n is the set of $\sigma \in S_n$ where $+$ or $-$ is placed on each integer. Let $pos(\sigma)$ and $neg(\sigma)$ count the $+$'s and $-$'s.

D_n is the subset of B_n where $neg(\sigma)$ is even.

Descents and rises in B_n use the linear order Θ:

$$1 <_\Theta \cdots <_\Theta n <_\Theta -n <_\Theta \cdots <_\Theta -1.$$

Ex. Let $\sigma = -3 \quad -2 \quad -6 \quad +5 \quad -1 \quad +4$

Then $pos(\sigma) = 2$, $neg(\sigma) = 4$, $des_B(\sigma) = 3$, and $ris_B(\sigma) = 3$.

Let

$$f_3(n) = \begin{cases} 1 & \text{if } n = 0 \\ u^n y(x-y)^{n-1} + v^n x(y-x)^{n-1} & \text{if } n \geq 1. \end{cases}$$

and $\xi^{f_3}(e_n) = \dfrac{(-1)^{n-1}}{n!} f_3(n)$ as usual.

Theorem.

$$n!\xi^{f_3}(h_n) = \sum_{\sigma \in B_n} u^{pos(\sigma)} v^{neg(\sigma)} x^{des_B(\sigma)} y^{ris_B(\sigma)}.$$

Proof. We have,

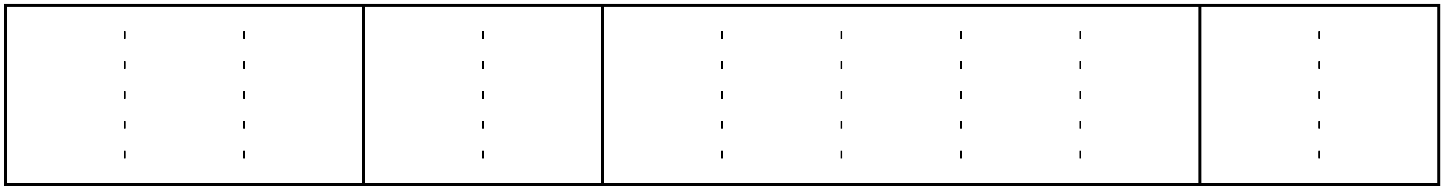
$$n!\xi^{f_3}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} (-1)^{\ell(\lambda)} f_3(\lambda_1) \cdots f_3(\lambda_\ell)$$

from which we create the following objects:

$$f_3(n) = u^n y(x-y)^{n-1} + v^n x(y-x)^{n-1}$$

$$n!\xi^{f_3}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} (-1)^{\ell(\lambda)} f_3(\lambda_1) \cdots f_3(\lambda_\ell)$$

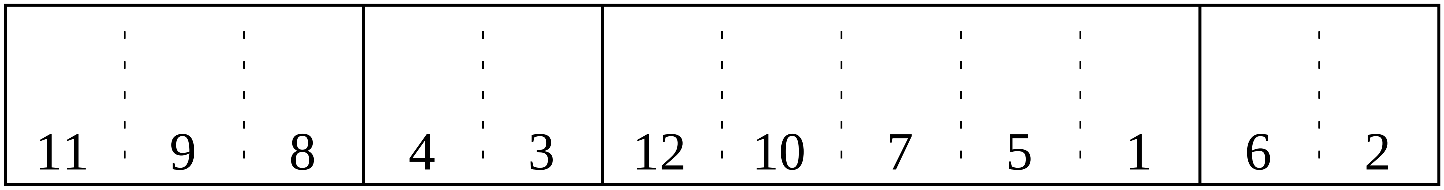
from which we create the following objects:



$$f_3(n) = u^n y(x-y)^{n-1} + v^n x(y-x)^{n-1}$$

$$n!\xi^{f_3}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell} B_{\lambda,n} f_3(\lambda_1) \cdots f_3(\lambda_\ell)$$

from which we create the following objects:



$$f_3(n) = u^n y(x-y)^{n-1} + v^n x(y-x)^{n-1}$$

$$n!\xi^{f_3}(h_n) = \sum_{\lambda\vdash n}\binom{n}{\lambda_1,\ldots,\lambda_\ell}B_{\lambda,n}f_3(\lambda_1)\cdots f_3(\lambda_\ell)$$

from which we create the following objects:

v	v	v	u	u	u	u	u	u	u	v	v
y	y	x	x	y	x	$-y$	$-y$	x	y	y	x
11	9	8	4	3	12	10	7	5	1	6	2

$$f_3(n) = u^n y(x-y)^{n-1} + v^n x(y-x)^{n-1}$$

Scan for the first instance of a "–" sign or two bricks with a decrease and the same u or v label. Break or combine accordingly.

In this way,

$$\begin{array}{|c:c:c|c:c|c:c:c:c:c|c:c|}
\hline
v & v & v & u & u & u & u & u & u & u & v & v \\
y & y & x & x & y & x & -y & -y & x & y & y & x \\
11 & 9 & 8 & 4 & 3 & 12 & 10 & 7 & 5 & 1 & 6 & 2 \\
\hline
\end{array}$$

is sent to

$$\begin{array}{|c:c:c|c:c|c:c|c:c:c|c:c|}
\hline
v & v & v & u & u & u & u & u & u & u & v & v \\
y & y & x & x & y & x & y & -y & x & y & y & x \\
11 & 9 & 8 & 4 & 3 & 12 & 10 & 7 & 5 & 1 & 6 & 2 \\
\hline
\end{array}$$

A fixed point:

v	v	v	u	u	u	u	u	u	u	v	v
y	y	x	x	y	x	x	x	x	y	y	x
11	9	8	4	3	12	10	7	5	1	6	2

Interpret every u as positive and v as negative. The above fixed point corresponds to

$$-11 \quad -9 \quad -8 \quad +4 \quad +3 \quad +12 \quad +10 \quad +7 \quad +5 \quad +1 \quad -6 \quad -2$$

Summing over all fixed points completes the proof. $\square$

The generating function:

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in B_n} u^{pos(\sigma)}v^{neg(\sigma)}x^{des_B(\sigma)}y^{ris_B(\sigma)} = \frac{x-y}{xe^{tv(y-x)}-ye^{tu(x-y)}}.$$

To find a generating function for D_n, notice that for $\sigma \in B_n$

$$\frac{v^{neg(\sigma)}+(-v)^{neg(\sigma)}}{2} = \begin{cases} v^{neg(\sigma)} & \text{if } \sigma \in D_n, \\ 0 & \text{if } \sigma \notin D_n. \end{cases}$$

Therefore,

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in D_n}u^{pos(\sigma)}v^{neg(\sigma)}x^{des_B(\sigma)}y^{ris_B(\sigma)}$$

$$= \quad \frac{x-y}{2xe^{tv(y-x)}-2ye^{tu(x-y)}}+\frac{x-y}{2xe^{tv(x-y)}-2ye^{tu(x-y)}}.$$

For $\sigma^1, \ldots, \sigma^m \in S_n$,

$comdes(\sigma^1, \ldots, \sigma^m)$ is the number of times $\sigma_i^j > \sigma_{i+1}^j$ for all j

$oneris(\sigma^1, \ldots, \sigma^m)$ is the number of times $\sigma_i^j < \sigma_{i+1}^j$ for at least one j

$$Ex.\ \text{If}\quad \begin{array}{rcllllllllllll} \sigma^1 & = & \textcolor{blue}{12} & \textcolor{blue}{10} & \textcolor{blue}{6} & \textcolor{blue}{4} & \textcolor{red}{3} & \textcolor{blue}{9} & \textcolor{red}{2} & \textcolor{blue}{11} & \textcolor{red}{1} & \textcolor{blue}{8} & \textcolor{red}{5} & \textcolor{red}{7} \\ \sigma^2 & = & \textcolor{blue}{10} & \textcolor{red}{5} & \textcolor{blue}{11} & \textcolor{red}{2} & \textcolor{blue}{6} & \textcolor{red}{1} & \textcolor{blue}{7} & \textcolor{red}{4} & \textcolor{blue}{12} & \textcolor{blue}{9} & \textcolor{blue}{8} & \textcolor{red}{3}, \\ \sigma^3 & = & \textcolor{blue}{6} & \textcolor{blue}{5} & \textcolor{blue}{12} & \textcolor{blue}{1} & \textcolor{red}{4} & \textcolor{red}{9} & \textcolor{blue}{10} & \textcolor{blue}{3} & \textcolor{red}{2} & \textcolor{blue}{11} & \textcolor{blue}{8} & \textcolor{red}{7} \end{array} \quad \text{then}$$

$comdes(\sigma^1, \sigma^2, \sigma^3) = 3$ and $oneris(\sigma^1, \sigma^2, \sigma^3) = 9$.

Let $m \geq 1$ and $g_1(n) = \dfrac{1}{(n!)^m}$ such that

$$\xi^{f_4}(e_n) \; = \begin{cases} 0 & \text{if } n = 0 \\ \frac{(-1)^{n-1}}{(n!)^m} y(x-y)^{n-1} & \text{if } n \geq 1 \end{cases}$$

Theorem.

$$(n!)^m \xi^{f_4}(h_n) = \sum_{\sigma \in S_n^m} x^{comdes(\sigma)} y^{oneris(\sigma)}.$$

Proof.

$$
\begin{aligned}
&(n!)^m \xi^{f_4}(h_n) \\
&= (n!)^m \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \xi^{f_4}(e_\lambda) \\
&= (n!)^m \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{\lambda_i - 1}}{(\lambda_i!)^m} y(x-y)^{\lambda_i - 1} \\
&= \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell}^m B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} y(x-y)^{\lambda_i - 1}.
\end{aligned}
$$

We have

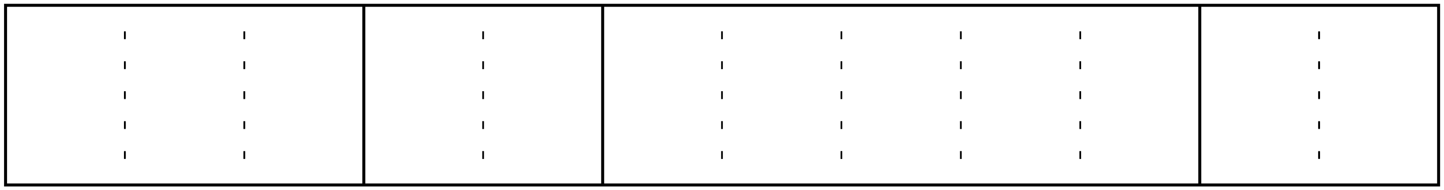
$$(n!)^m \xi^{f_4}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell}^m B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} y(x-y)^{\lambda_i - 1}.$$

which may be used to create:

We have

$$(n!)^m \xi^{f_4}(h_n) = \sum_{\lambda\vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell}^m B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} y(x-y)^{\lambda_i - 1}.$$

which may be used to create:



We have

$$(n!)^m \xi^{f_4}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell}^m B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} y(x-y)^{\lambda_i - 1}.$$

which may be used to create:

11	9	3	12	2	10	6	5	4	1	8	7
12	8	5	11	10	9	7	6	4	3	2	1

We have

$$(n!)^m \xi^{f_4}(h_n) = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \ldots, \lambda_\ell}^m B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} y(x-y)^{\lambda_i - 1}.$$

which may be used to create:

x	x	y	x	y	$-y$	x	$-y$	$-y$	y	x	y
11	9	3	12	2	10	6	5	4	1	8	7
12	8	5	11	10	9	7	6	4	3	2	1

Apply the brick breaking/combining involution where bricks are combined if two bricks have a common descent between them. Then

x	x	$-y$	x	y	$-y$	x	$-y$	$-y$	y	x	y
12	11	9	3	2	10	6	5	4	1	8	7
12	11	10	8	5	9	7	6	4	3	2	1

is sent to

x	x	y	x	y	$-y$	x	$-y$	$-y$	y	x	y
12	11	9	3	2	10	6	5	4	1	8	7
12	11	10	8	5	9	7	6	4	3	2	1

A fixed point:

x	x	y	x	y	x	x	x	x	y	x	y
11	9	3	12	2	10	6	5	4	1	8	7
12	8	5	11	10	9	7	6	4	3	2	1

This fixed point corresponds to:

$$\begin{aligned}\sigma^1 &= 11\quad 9\quad 3\quad 12\quad 2\quad 10\quad 6\quad 5\quad 4\quad 1\quad 8\quad 7\\ \sigma^2 &= 12\quad 8\quad 5\quad 11\quad 10\quad 9\quad 7\quad 6\quad 4\quad 3\quad 2\quad 1\end{aligned}$$

Summing all fixed points gives the desired result. □

The generating function which follows:

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{(n!)^m}\sum_{\sigma\in S_n^m} x^{comdes(\sigma)}y^{oneris(\sigma)} &= \left(1+\sum_{n\geq 1}(-t)^n\frac{(-1)^{n-1}}{(n!)^m}y(x-y)^{n-1}\right)^{-1}\\ &= \frac{x-y}{x-y\sum_{n\geq 0}\frac{(x-y)^n t^n}{(n!)^m}}.\end{aligned}$$

For $\sigma_1 \cdots \sigma_n \in S_n$,

$inv(\sigma)$ is the number of times $\sigma_i > \sigma_j$ for $i < j$

$coinv(\sigma)$ is the number of times $\sigma_i < \sigma_j$ for $i < j$

Let $[n]_{p,q} = \dfrac{p^n - q^n}{p - q}$ and $[n]_{p,q}! = [n]_{p,q} \cdots [2]_{p,q}[1]_{p,q}$.

Let

$$\begin{bmatrix} n \\ \lambda_1, \ldots, \lambda_\ell \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[\lambda_1]_{p,q}! \cdots [\lambda_\ell]_{p,q}!}$$

Let $\mathcal{R}(1^{\lambda_1}, \ldots, \ell^{\lambda_\ell})$ be the set of rearrangements of λ_1 1's, λ_2 2's,...

A generalization of a theorem of Carlitz:

$$\begin{bmatrix} n \\ \lambda_1, \ldots, \lambda_\ell \end{bmatrix}_{p,q} = \sum_{r \in \mathcal{R}(1^{\lambda_1}, \ldots, \ell^{\lambda_\ell})} q^{inv(r)} p^{coinv(r)}.$$

Given a rearrangement r of b_1 1s, b_2 2s, . . . , b_l l's, form a permutation $\sigma(r)$ by labelling the 1s in r from right to left with 1, 2, . . . , b_1. Then number the 2s from right to left with $b_1 + 1, b_1 + 2, \ldots, b_1 + b_2$, and so on.

For example, let $b_1 = 4, b_2 = 2, b_3 = 3$ and

$$\begin{aligned} r &= 1\,3\;\;2\,1\;\;3\,3\;\;1\,2\,1\,3\,3 \\ \sigma(r) &= 4\,11\,6\,3\,10\,9\,2\,5\,1\,8\,7 \end{aligned}$$

$$\sigma(r)^{-1} = 9\,7\,4\,1\;\;8\,3\;\;11\,10\,6\,5\,2$$

$inv(\sigma(r)) = inv(r) +$ the number of inversions introduced by changing the 1s in r to $1, 2, \ldots, b_1$, the $2s$ to $b_1 + 1, b_1 + 2, \ldots, b_1 + b_2$, and so on.

The number of inversions introduced by changing the 1s to $1, 2, \ldots, b_1$ is $(b_1 - 1) + (b_1 - 2) + \cdots + 1 = \binom{b_1}{2}$.

$$\mathrm{inv}(\sigma(r)) = \mathrm{inv}(r) + \sum_k \binom{b_k}{2}$$

$$\mathrm{coinv}(\sigma(r)) = \mathrm{coinv}(r)$$

Theorem 0.1. *If* $b_1 + b_2 + \cdots + b_l = n$ *where each* b_i *is a positive integer, then*

$$\begin{bmatrix} n \\ b_1, b_2, \ldots, b_l \end{bmatrix}_{p,q} q^{\sum_i \binom{b_i}{2}} = \sum_{\tau \in \mathrm{dec}_n(b_1,\ldots,b_l)} q^{\mathrm{inv}(\tau)} p^{\mathrm{coinv}(\tau)}$$

where $\mathrm{dec}_n(b_1, ..., b_l)$ *is the set of permutations in* S_n *that when read from left to right in one-line notation consist of decreasing sequences in blocks of size* $b_1, b_2, \ldots, b_l$.

$$f_1(n) = \begin{cases} 1 & \text{if } n = 0 \\ y\,(x-y)^{n-1} & \text{if } n \geq 1. \end{cases}$$

Define $\xi^{f_1} : \Lambda \longrightarrow \mathbb{Q}[x,y]$ as a homomorphism such that

$$\xi^{f_1}_{p,q}(e_n) = \frac{(-1)^{n-1} q^{\binom{n}{2}}}{[n]_{p,q}!} f_1(n).$$

Theorem.

$$[n]_{p,q}!\,\xi^{f_1}_{p,q}(h_n) = \sum_{\sigma \in S_n} x^{des(\sigma)} y^{ris(\sigma)} q^{inv(\sigma)} p^{coinv(\sigma)}.$$

Proof.

$[n]_{p,q}!\xi_{p,q}^{f_1}(h_n)$

$$= \quad [n]_{p,q}!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\xi_{p,q}^{f_1}(e_\lambda)$$

$$= \quad \sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda_1,\ldots,\lambda_\ell \end{bmatrix}_{p,q} q^{\binom{\lambda_1}{2}+\cdots+\binom{\lambda_\ell}{2}}B_{\lambda,n}(-1)^{\ell(\lambda)-1}f_1(\lambda_1)\cdots f_1(\lambda_\ell).$$

$f_1(n) = y(x-y)^{n-1}$

$$\sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda_1,\ldots,\lambda_\ell\end{bmatrix}_{p,q} q^{\binom{\lambda_1}{2}+\cdots+\binom{\lambda_\ell}{2}} B_{\lambda,n} f_1(\lambda_1)\cdots f_1(\lambda_\ell).$$

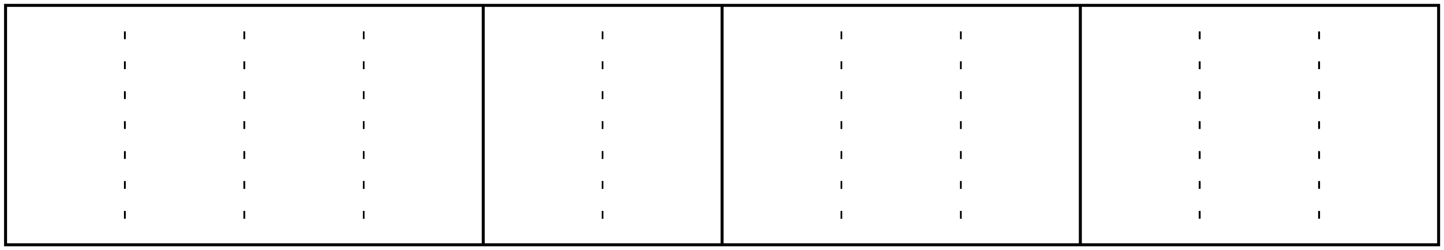
which may be used to create:

$$f_1(n) = y(x-y)^{n-1}$$

The term we have to work with is

$$\sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda_1,\ldots,\lambda_\ell\end{bmatrix}_{p,q} q^{\binom{\lambda_1}{2}+\cdots+\binom{\lambda_\ell}{2}} B_{\lambda,n} f_1(\lambda_1)\cdots f_1(\lambda_\ell).$$

which may be used to create:



$f_1(n) = y(x-y)^{n-1}$

$$\sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda_1,\dots,\lambda_\ell\end{bmatrix}_{p,q} q^{\binom{\lambda_1}{2}+\cdots+\binom{\lambda_\ell}{2}} B_{\lambda,n} f_1(\lambda_1)\cdots f_1(\lambda_\ell).$$

which may be used to create:

q^{11}	q^7	q^6	q^4	q^6	q^5	q^5	q^4	q^2	q^2	q^1	q^0
p^0	p^3	p^3	p^4	p^1	p^1	p^0	p^0	p^1	p^0	p^0	p^0
12	8	7	5	10	9	11	6	3	4	2	1

$$f_1(n) = y(x-y)^{n-1}$$

The term we have to work with is

$$\sum_{\lambda\vdash n}\begin{bmatrix} n \\ \lambda_1,\ldots,\lambda_\ell\end{bmatrix}_{p,q} q^{\binom{\lambda_1}{2}+\cdots+\binom{\lambda_\ell}{2}} B_{\lambda,n} f_1(\lambda_1)\cdots f_1(\lambda_\ell).$$

which may be used to create:

x	x	$-y$	y	x	y	x	$-y$	y	$-y$	$-y$	y
q^{11}	q^7	q^6	q^4	q^6	q^5	q^5	q^4	q^2	q^2	q^1	q^0
p^0	p^3	p^3	p^4	p^1	p^1	p^0	p^0	p^1	p^0	p^0	p^0
12	8	7	5	10	9	11	6	3	4	2	1

$$f_1(n) = y(x-y)^{n-1}$$

Apply the brick breaking/combining involution so that

x	x	$-y$	y	x	y	x	$-y$	y	$-y$	$-y$	y
q^{11}	q^7	q^6	q^4	q^6	q^5	q^5	q^4	q^2	q^2	q^1	q^0
p^0	p^3	p^3	p^4	p^1	p^1	p^0	p^0	p^1	p^0	p^0	p^0
12	8	7	5	10	9	11	6	3	4	2	1

is sent to

x	x	y	y	x	y	x	$-y$	y	$-y$	$-y$	y
q^{11}	q^7	q^6	q^4	q^6	q^5	q^5	q^4	q^2	q^2	q^1	q^0
p^0	p^3	p^3	p^4	p^1	p^1	p^0	p^0	p^1	p^0	p^0	p^0
12	8	7	5	10	9	11	6	3	4	2	1

A fixed point under this involution:

x	x	x	y	x	y	x	x	y	x	x	y
q^{11}	q^{7}	q^{6}	q^{4}	q^{6}	q^{5}	q^{5}	q^{4}	q^{2}	q^{2}	q^{1}	q^{0}
p^{0}	p^{3}	p^{3}	p^{4}	p^{1}	p^{1}	p^{0}	p^{0}	p^{1}	p^{0}	p^{0}	p^{0}
12	8	7	5	10	9	11	6	3	4	2	1

The powers of x, y, q and p on a fixed point register the correct statistics.

Summing all fixed points completes the proof. $\square$

The generating function which follows:

$$\sum_{n\geq 0} \frac{t^n}{[n]!} \sum_{\sigma\in S_n} x^{des(\sigma)} y^{ris(\sigma)} q^{inv(\sigma)} p^{coinv(\sigma)} = \frac{x-y}{x - y\exp_{q,p}^{t(x-y)}}$$

where for a series $r(t) = \sum_{n\geq 0} a_n \frac{t^n}{n!}$, we let $r_{q,p}(t) = \sum_{n\geq 0} a_n \frac{t^n}{[n]_{p,q}!} q^{\binom{n}{2}}$.

Replacing $\frac{(-1)^{n-1}}{n!}$ by $\frac{(-1)^{n-1}q^{\binom{n}{2}}}{[n]_{p,q}!}$ in our other proofs will allow us to prove the following.

$$\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{\substack{\sigma\in S_n\\ \sigma \text{ is even alt.}}} q^{inv(\sigma)}p^{coinv(\sigma)} = \frac{1}{\cos_{q,p}(t)}$$

and

$$\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{\sigma\in B_n} u^{pos(\sigma)}v^{neg(\sigma)}x^{des_B(\sigma)}y^{ris_B(\sigma)}q^{inv(\sigma)}p^{coinv(\sigma)}$$

$$= \frac{x-y}{xe_{q,p}^{tv(y-x)} - ye_{q,p}^{tu(x-y)}}.$$