Generating functions from symmetric functions

Anthony Mendes and Jeffrey Remmel

Abstract. This monograph introduces a method of finding and refining generating functions. By manipulating combinatorial objects known as brick tabloids, we will show how many well known generating functions may be found and subsequently generalized. New results are given as well.

The techniques described in this monograph originate from a thorough understanding of a connection between symmetric functions and the permutation enumeration of the symmetric group. Define a homomorphism ξ on the ring of symmetric functions by defining it on the elementary symmetric function e_n such that $\xi(e_n) = (1-x)^{n-1}/n!$. Brenti showed that applying ξ to the homogeneous symmetric function gave a generating function for the Eulerian polynomials [**14, 13**].

Beck and Remmel reproved the results of Brenti combinatorially [**6**]. A handful of authors have tinkered with their proof to discover results about the permutation enumeration for signed permutations and multiples of permutations [**4, 5, 51, 52, 53, 58, 70, 71**]. However, this monograph records the true power and adaptability of this relationship between symmetric functions and permutation enumeration. We will give versatile methods unifying a large number of results in the theory of permutation enumeration for the symmetric group, subsets of the symmetric group, and assorted Coxeter groups, and many other objects.

Contents

CHAPTER 1

Brick tabloids in permutation enumeration

We will begin with a discussion on the ring of formal power series in order to establish basic definitions and concepts needed understand the rest of this work. Generating functions and the benefits of their use are introduced here. The objectives of this monograph and a brief introduction to permutation statistics are also included in Section 1.1.

Symmetric function theory will be used heavily throughout this work. In Section 1.2, the ring of symmetric functions is described completely from scratch. Our approach is not like that of any other published work in that we attempt to give combinatorial proofs for every basic symmetric function identity. It is in Section 1.2 where we introduce the notion of brick tabloids, a combinatorial object used extensively in the rest of this monograph.

Section 1.3 contains the ideas at the heart of this monograph. Symmetric function theory is used to create generating functions here. The rest of this monograph is devoted to further developing the methods in Section 1.3—this section is essential reading for those wanting a complete understanding of our building of generating functions.

We systematically compare all previous published works relating brick tabloids and permutation statistics with the results in this monograph in Section 1.4. This should help to clarify the advances we will make in this work.

Many proofs in the first three sections have been revamped and elements of the combinatorial approach to basic symmetric function identities included in this section have not been previously published, but this content is well known and none of it should be considered brand new. This is not the case for Section 1.5, however. Although some of the generating functions in this section are known, the methodology is original. It is in this section where we make our first significant advances in our development when we discuss ways to extend the homomorphism introduced in Section 1.3 to find and refine more results about descents in the symmetric group.

1.1. The ring of formal power series

Let $a_0, a_1, \ldots$ be a sequence in a ring R and t an indeterminate. The formal power series for the sequence $a_0, a_1, \ldots$ is the expression

$$a_0t^0 + a_1t^1 + a_2t^2 + \cdots .$$

Such an object may be denoted $\sum_{n=0}^{\infty} a_n t^n$. At face value, although we are using the plus symbol and summation notation, we are not performing the operation of addition. We are simply presenting the sequence $a_0, a_1, \ldots$ in a specific way, using plus symbols to separate terms and using powers of t as placeholders. The set of all formal power series in t representing sequences in the ring R will be denoted by $R[[t]]$.

For instance, consider the sequence in the ring of rational numbers defined by $a_n = 2^{n(n-1)/2}/n!$. This sequence recorded as a formal power series is

$$1 + t + t^2 + \frac{8}{6}t^3 + \frac{64}{24}t^4 + \cdots = \sum_{n=0}^{\infty} \frac{2^{n(n-1)/2}}{n!} t^n. \tag{1.1}$$

If interpreted as a complex-valued function in the variable t, the above series would have a radius of convergence of 0 and thus most techniques from analysis would not apply. The object in (1.1) does, however, define a perfectly valid formal power series. In fact, this formal power series will be used in Section 3.4 to count the number of graphs on n nodes.

If $a_0, a_1, \ldots$ is a sequence in R such that $a_0 = a_1 = \cdots = a_{j-1} = 0$ for some $j \geq 0$, then we may denote $\sum_{n=0}^{\infty} a_n t^n$ as $\sum_{n=j}^{\infty} a_n t^n$ to reflect this fact. Along similar lines, we write $\sum_{n=0}^{j} a_n t^n$ provided $a_n = 0$ for $n > j$.

There are a handful of operations on $R[[t]]$ which we now define. First, for each nonnegative integer j, define a map $\cdot|_{t^j}$ from $R[[t]]$ to R such that

$$\left. \sum_{n=0}^{\infty} a_n t^n \right|_{t^j} = a_j.$$

The element in R found by an application of $\cdot|_{t^j}$ is called the coefficient of t^j. Two elements in $R[[t]]$ are equal provided the coefficients of t^j in each formal power series are equal for all $j \geq 0$.

Define the sum of two formal power series by the rule

$$\left(\sum_{n=0}^{\infty} a_n t^n \right) + \left(\sum_{n=0}^{\infty} b_n t^n \right) = \sum_{n=0}^{\infty} (a_n + b_n) t^n$$

where the plus symbol on the right hand side of the equation denotes the sum of two elements in the ring R. Define the product of two formal power series by the rule

$$\left(\sum_{n=0}^{\infty} a_n t^n \right) \left(\sum_{n=0}^{\infty} b_n t^n \right) = \sum_{n=0}^{\infty} \left(a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 \right) t^n$$

where the plus symbols and the adjacent elements on the right hand side of the above equation denote the sum and product of two elements in R, respectively. With these definitions, it is not difficult to show that $R[[t]]$ is a ring; naturally, $R[[t]]$ is called the ring of formal power series. Notice that R is commutative if and only if $R[[t]]$ is commutative. For greater simplicity in our development of the ring of power series, we make the assumption that R is a commutative ring with unity from now on.

If

$$\left(\sum_{n=0}^{\infty} a_n t^n \right) \left(\sum_{n=0}^{\infty} b_n t^n \right) = 1$$

where 1 represents $1 + 0t + 0t^2 + \cdots$, then we say $\sum_{n=0}^{\infty} b_n t^n$ is the reciprocal of $\sum_{n=0}^{\infty} a_n t^n$ and write

$$\sum_{n=0}^{\infty} b_n t^n = \left(\sum_{n=0}^{\infty} a_n t^n\right)^{-1} = \frac{1}{\sum_{n=0}^{\infty} a_n t^n}.$$

For example, by our definition of the product of two formal power series, $(1-t)(1+t+t^2+\cdots) = 1$ and therefore $(1+t+t^2+\cdots) = (1-t)^{-1} = 1/(1-t)$. Formally, this is the familiar formula for the sum of a geometric series.

Define the composition of $\sum_{n=0}^{\infty} a_n t^n$ and $\sum_{m=1}^{\infty} b_m t^m$ as the formal power series

$$\sum_{n=0}^{\infty} a_n \left(\sum_{m=1}^{\infty} b_m t^m\right)^n.$$

A potential problem in this definition arises if any coefficient in the above formal power series is an infinite sum of elements in R. However, notice that

$$\begin{aligned} \left.\sum_{n=0}^{\infty} a_n \left(\sum_{m=1}^{\infty} b_m t^m\right)^n\right|_{t^j} &= \left.\sum_{n=0}^{\infty} a_n \left(b_1 t^1 + \cdots + b_j t^j\right)^n\right|_{t^j} \\ &= \left.\sum_{n=0}^{j} a_n \left(b_1 t^1 + \cdots + b_j t^j\right)^n\right|_{t^j} \end{aligned}$$

where in the last expression we are selecting the coefficient of t^j in a finite sum. This shows that the stipulation $b_0 = 0$ forces the coefficient of t^j in the composition as defined above to be a finite sum of elements in R for $j \geq 0$. Thus, there are no problems with our definition of composition.

The derivative of a formal power series is a map $d/dt(\cdot)$ from $R[[t]]$ to $R[[t]]$ defined by

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty} a_n t^n\right) = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

where $n+1$ is the element $1 + \cdots + 1$ ($n+1$ times) in R. Similarly, the integral of a formal power series is a map $\int \cdot \, dt$ from $R[[t]]$ to $R[[t]]$ defined by

$$\int \left(\sum_{n=0}^{\infty} a_n t^n\right) dt = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} t^n$$

provided the multiplicative inverse of n exists in R for $n \geq 1$. Notice that by our definition of integration, the coefficient of t^0 in the integral of any formal power series is defined to be 0. The derivative and integral for formal power series obey many of the same laws as the differentiation and integration of complex-valued functions. For example, it may be shown without much effort that the product rule, chain rule, and quotient rule all hold for formal power series.

Let R_1, R_2 be rings and $\xi : R_1 \to R_2$ a ring homomorphism. The map ξ may be considered a ring homomorphism from $R_1[[t]]$ to $R_2[[t]]$ by letting

$$\xi\left(\sum_{n=0}^{\infty} a_n t^n\right) = \sum_{n=0}^{\infty} \xi(a_n) t^n.$$

Suppose that two formal power series $\sum_{n=0}^{\infty} a_n t^n$ and $\sum_{n=0}^{\infty} b_n t^n$ in $R_1[[t]]$ are reciprocals of one another. In this case, since $\sum_{m=0}^{n} a_m b_{n-m} = 0$ for $n \geq 1$, we have that

$$0 = \xi(0) = \xi\left(\sum_{m=0}^{n} a_m b_{n-m}\right) = \sum_{m=0}^{n} \xi(a_m)\xi(b_{n-m})$$

for $n \geq 1$. Therefore, $\xi\left(\sum_{n=0}^{\infty} a_n t^n\right)$ and $\xi\left(\sum_{n=0}^{\infty} b_n t^n\right)$ are reciprocals of one another. This gives

$$\xi\left(\sum_{n=0}^{\infty} a_n t^n\right)^{-1} = \left(\sum_{n=0}^{\infty} \xi(a_n) t^n\right)^{-1}.$$

Thus, homomorphisms interact nicely with the operation of taking reciprocals (as well as most other operations on the ring of formal power series). These type of manipulations involving homomorphisms on the ring of formal power series will be used many times throughout this monograph.

Even though elements in $R[[t]]$ are not functions of t, our definitions for the coefficient, sum, product, reciprocal, composition, derivative, and integral all behave as if they were. That is, when our formal power series are interpreted as complex-valued functions in the variable t, every one of our definitions is the natural one. For this reason, if we encounter a formal power series which may be interpreted as a named complex-valued function, then we will use that name in reference to the formal power series. For instance, if R is the ring of rational numbers, the formal power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n}$$

will be referred to as $\cos(t)$. Note that $\cos(t)$ is only a nickname for the formal power series displayed above. However, since all of the operations we have defined are true within the radius of convergence for these complex-valued functions, it is usually safe to treat these formal power series as functions.

The formal power series for the sequence $a_0, a_1, \ldots$ is commonly referred to as the generating function for $a_0, a_1, \ldots$. We have intentionally waited to introduce this terminology until after our description of $R[[t]]$ to avoid any potential confusion in reference to the word "function" but adopt it for the rest of this document.

Generating functions are usually the preferred way to investigate the properties of a given sequence and since the time of Euler and Laplace they have become a standard tool to the combinatorialist. Some of the benefits of finding a generating function include the following.

- Generating functions can give the n^{th} term of a sequence when simple, direct formulas may not exist. There are methods to extract the coefficient of formal power series which are independent of the notion of convergence.
- Averages, variances, and other statistical properties of a sequence may be rapidly calculated.
- When viewed as a function of a complex variable where convergent, the asymptotic properties of a sequence may be found using elementary complex analysis.
- Sometimes, symmetric, unimodal, and convex properties of sequences may be found with the help of generating functions.

In short, generating functions provide convenient ways to manipulate sequences when other methods can be unwieldy. They give a fundamental understanding of a sequence like nothing else.

The goal of this monograph is to introduce a new, unifying technique of finding generating functions. A myriad of new and well known results may be found with the method we will describe.

This work is not the first attempt at consolidating the patchwork of known ways of finding generating functions. The exponential formula explains where an assortment of them come from. Stanley has shown how to find generating functions by understanding the incidence algebra of partially ordered sets with certain nice properties [**65, 66**]. Linked sets, together with many examples of their use, were introduced in Gessel's thesis [**42**]. Furthermore, Jackson and Aleliunas have given a nice theory of finding generating functions by decomposing sequences into their maximal paths [**45**].

Through stated theorems or implicitly through Möbius functions, the techniques via partially ordered sets, linked sets, or maximal paths all share the common theme of reciprocation. These three works indicate that it is often easier to find the reciprocal of a generating function than to find it directly. In the same vein, the underlying framework for our ideas is a combinatorial understanding of division. With this understanding, the generating functions in the works of Stanley, Gessel, and Jackson and Aleliunas can be found. The exponential formula can also be proved in the ways we describe.

A particularly nice aspect of the methods introduced here is the ability to go "backward". Suppose we want to prove that a given function is the generating function for a certain sequence. It may be possible to extract information from the function in order to prove, combinatorially, that actually is the desired generating function. Many times, after this combinatorial proof is found, generalizations to the result are immediate.

Guiding examples for us will come from the study of permutation statistics. A permutation statistic is not a statistic in the strictest sense, but rather a function mapping permutations to nonnegative integers. The modern analysis of such objects began in the early twentieth century with the work of MacMahon [**55**]. He refined the "classic" notions of the descent, excedance, inversion, and major index statistics. They are defined such that if $\sigma = \sigma_1 \cdots \sigma_n$ is an element of the symmetric group S_n written in one line notation, then

$$\mathrm{des}(\sigma) = \sum_{i=1}^{n-1} \chi(\sigma_{i+1} < \sigma_i), \qquad \mathrm{exc}(\sigma) = \sum_{i=1}^{n} \chi(i < \sigma_i),$$

$$\mathrm{maj}(\sigma) = \sum_{i=1}^{n-1} i\,\chi(\sigma_{i+1} < \sigma_i), \qquad \text{and} \qquad \mathrm{inv}(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j),$$

where for any statement A, $\chi(A)$ is 1 if A is true and 0 if A is false. [†] Below we have provided a table of the values of these permutation statistics for the symmetric

[†]In addition to those elements in the symmetric group, these definitions hold for any finite sequence of numbers.

group S_3.

σ	$\mathrm{des}(\sigma)$	$\mathrm{exc}(\sigma)$	$\mathrm{inv}(\sigma)$	$\mathrm{maj}(\sigma)$
1 2 3	0	0	0	0
1 3 2	1	1	1	2
2 1 3	1	1	1	1
2 3 1	1	2	2	2
3 1 2	1	1	2	1
3 2 1	2	1	3	3

A variety of different permutation statistics will be defined throughout this work. A listing is available in an appendix. Note that the first two and the last two columns of the table are equidistributed, i.e. they have the same number of 0's, 1's, 2's, and 3's.

Properties of subsequent generalizations of these statistics—together with many new statistics for various permutation groups—remain an active area of research today. In the past few decades, beautiful combinatorial and bijective proofs of classical and new results have been published (one of the first along these lines proves that the inversion and major index statistics are equidistributed over the symmetric group [**36**]). An abundance of papers are devoted to the relationship between generating functions and permutation statistics.

Let $\mathbb{Q}[x_1, \ldots, x_N]$ be the ring of polynomials in the variables $x_1, \ldots, x_N$ with coefficients in $\mathbb{Q}$. Most of the generating functions in this monograph are elements in $\mathbb{Q}[x_1, \ldots, x_N][[t]]$ which register permutation statistics over subsets of the symmetric group. For instance, let $E_0(x) = 1$ and $E_n(x) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)+1}$ be the so-called Eulerian polynomials. They look like

$$
\begin{aligned}
E_0(x) &= 1\\
E_1(x) &= x\\
E_2(x) &= x + x^2\\
E_3(x) &= x + 4x^2 + x^3\\
E_4(x) &= x + 11x^2 + 11x^3 + x^4\\
E_5(x) &= x + 26x^2 + 66x^3 + 26x^4 + x^5\\
E_6(x) &= x + 57x^2 + 302x^3 + 302x^4 + 57x^5 + x^6\\
&\vdots
\end{aligned}
$$

The closed expression

$$\sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n = \frac{1-x}{1 - x\exp(t(1-x))}$$

is implicit in our Theorem 1.17. Results like this, which give generating functions involving polynomials which keep track of permutation statistics, are indicative of those which permeate this work.

Although all of our generating functions will come from understanding the reciprocation of power series combinatorially, the language of symmetric functions will prove useful. Therefore, we commence with a brief description of the ring of symmetric functions. Like the theory of generating functions, this is a vast and beautiful area of mathematics, and therefore we cannot provide a full account of the theory of symmetric functions. Section 1.2 contains only those results pertinent to our developing theory. Included are a few previously unpublished combinatorial proofs of fundamental identities.

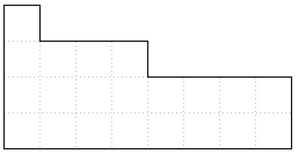
1.2. The ring of symmetric functions

This section contains an exposition on the ring of symmetric functions. Basic definitions and notation are established so that our later development of building generating functions may be understood. We will use combinatorial proofs whenever possible.

A partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a finite sequence of weakly increasing nonnegative integers. Partitions index the conjugacy classes of the symmetric group as well as the elements in a basis for the ring of symmetric functions. Therefore, partitions will be used extensively throughout our work.

We will let $|\lambda|$ be the sum of the integers in the partition λ and $\ell(\lambda)$ be the length of λ. If $|\lambda| = n$, then we say λ is a partition of n and write $\lambda \vdash n$. A partition may be denoted $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ where m_i is the number of parts in λ equal to i. With this notation, the number z_λ is equal to $1^{m_1} \cdots n^{m_n} m_1! \cdots m_n!$.

A partition λ is identified with its (French) Ferrers diagram which is $\ell(\lambda)$ rows of left justified squares where the i^{th} row has length λ_i reading top to bottom. The Ferrers diagram for $(1, 4, 8, 8)$ is below.



Squares in a Ferrers diagram will be referred to as cells. The conjugate partition to λ, denoted λ', is the partition obtained by flipping the Ferrers diagram of λ so that each column becomes a row and each row a column.

Partitions are sorted by the reverse lexicographic order so that if $\lambda, \mu \vdash n$, $\lambda \preceq \mu$ if the largest part of λ is greater than the largest part of μ. If the largest part of λ is the same as that in μ, then examine the second largest part of λ and μ, etc. The partitions of 5 in reverse lexicographic order are

$$(5), (1, 4), (2, 3), (1^2, 3), (1, 2^2), (1^3, 2), (1^5).$$

See [**3**] for a detailed exposition on the theory of partitions.

Let CS_λ be the set of Ferrers diagrams for the partition λ where each cell in the diagram contains a positive integer such that the integers strictly increase within columns and weakly increase within rows. This is known as the set of column strict tableaux of shape λ. Given $T \in CS_\lambda$, let $T_{(i,j)}$ be the integer in cell (i, j) and define the weight of T, $w(T)$, such that

$$w(T) = \prod_{(i,j) \in T} x_{T_{(i,j)}}$$

for variables $x_1, x_2, \ldots$. Below we have included an example of all possible column strict tableaux of shape $(1, 2, 4)$ with weight $x_1^3 x_2^2 x_3 x_4$.

$$\begin{array}{cccc} 3 & & & \\ 2 & 2 & & \\ 1 & 1 & 1 & 4 \end{array} \qquad \begin{array}{cccc} 4 & & & \\ 2 & 2 & & \\ 1 & 1 & 1 & 3 \end{array} \qquad \begin{array}{cccc} 3 & & & \\ 2 & 4 & & \\ 1 & 1 & 1 & 2 \end{array} \qquad \begin{array}{cccc} 4 & & & \\ 2 & 3 & & \\ 1 & 1 & 1 & 2 \end{array}$$

A symmetric polynomial p in the variables $x_1, \dots, x_N$ is a polynomial with the property that

$$p(x_1, \dots, x_N) = p(x_{\sigma_1}, \dots, x_{\sigma_N})$$

for all $\sigma = \sigma_1 \cdots \sigma_N \in S_N$. Let Λ^N be the ring of symmetric polynomials in $x_1, \dots, x_N$ and Λ_n^N be the subset of Λ^N containing the homogeneous elements of degree n. Using the surjective ring homomorphism from Λ_n^{N+1} to Λ_n^N defined by taking $x_{N+1} = 0$, let $\Lambda_n = \varprojlim_N \Lambda_n^N$ for each $n \geq 0$. Define $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ to be the ring of symmetric functions.

Our technical definition of the ring of symmetric functions is needed to ensure the validity of taking an infinite series of monomials in an infinite number of variables, but symmetric functions may be thought of in a much simpler manner. In short, a symmetric function in the variables $x_1, x_2, \dots$ may be thought of as a symmetric polynomial in an infinite number of variables. For instance,

$$x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_2x_3 + x_2x_4 + x_2x_5 + \cdots + x_3x_4 + x_3x_5 + x_3x_6 + \cdots$$

is a symmetric function in Λ_2.

For $\lambda \vdash n$, the monomial symmetric function m_λ is the element in Λ_n given by the sum of all monomials where the exponents on the powers of x_i give a rearrangement of the parts of λ. For example, $m_{(2,1)}$ in 3 variables (meaning that $x_4 = x_5 = \cdots = 0$) is given below:

$$m_{(1,2)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2.$$

It is not difficult to see that any symmetric function where every term is of degree n must be a sum of monomial symmetric functions; therefore, $\{m_\lambda : \lambda \vdash n\}$ is a basis for Λ_n. This implies that the dimension of Λ_n is the number of partitions of n.

The elementary symmetric function e_n may be defined by using a formal power series in $\Lambda[[t]]$. Let

$$\sum_{n=0}^{\infty} e_n t^n = \prod_i (1 + x_i t). \tag{1.2}$$

Let $E(t)$ be the sum on the left hand side of the above equation. Since only one power of x_i may contribute to the coefficient of t^n on the left hand side of (1.2) for every i, e_n is the sum of all square free monomials in the variables $x_1, \dots, x_N$. For example,

$$e_3(x_1, x_2, x_3, x_4) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4.$$

It follows that the product in (1.2) is equal to $\sum w(T)$ where the sum runs over all possible $T \in CS_{(1^n)}$. For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. Later on, as a corollary to Theorem 1.10, we will show that the elementary symmetric functions form a basis for Λ_n. (This fact will not be used until after Theorem 1.10 so that this document is completely self contained.)

The homogeneous symmetric function h_n is defined such that

$$\sum_{n=0}^{\infty} h_n t^n = \prod_i \frac{1}{1-x_i t}. \tag{1.3}$$

Let $H(t)$ be the sum on the left hand side of the above equation. For example, h_3 in 3 variables is given below:

$h_3(x_1,x_2,x_3) = x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_1^3 + x_2^3 + x_3^3.$

Just as the n^{th} elementary symmetric function has a combinatorial interpretation in terms of column strict tableaux, so does the n^{th} homogeneous symmetric function. The series expansion of each term in the product in (1.3) gives that $h_n = \sum w(T)$ where the sum runs over all $T \in CS_{(n)}$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$. The fact that $\{h_\lambda : \lambda \vdash n\}$ is a basis for Λ_n will be a corollary to Theorem 1.12.

The definitions of the homogeneous and elementary symmetric functions give

$$\sum_{n=0}^{\infty} h_n t^n = H(t) = (E(-t))^{-1} = \left(\sum_{n=0}^{\infty} e_n(-t)^n\right)^{-1}. \tag{1.4}$$

This trivial restatement of definitions holds an astounding wealth of information about the permutation statistics for the symmetric group, subsets of the symmetric group, cross product groups, wreath product groups, linear recurrence equations with constant coefficients, and more. Through the machinery we will build, (1.4) will yield many generating functions.

Another basic fact as a result of these definitions is Lemma 1.1 below.

LEMMA 1.1. *For $n \geq 1$,*

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0.$$

PROOF 1. Compare the coefficient of t^n on both sides of

$$1 = H(t)E(-t) = \left(\sum_{n=0}^{\infty} h_n t^n\right)\left(\sum_{i=0}^{\infty} (-1)^i e_i t^i\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} (-1)^i e_i h_{n-i}\right) t^n. \quad \square$$

We now provide a second, combinatorial proof of Lemma 1.1 which will be used again to help with the proof of Lemma 1.7.

PROOF 2. Let $\mathcal{T}$ be the set of all pairs (T,S) such that $T \in CS_{(1^i)}$ and $S \in CS_{(n-i)}$ where $0 \leq i \leq n$ (the hypothesis that $n \geq 1$ forces one of T or S to be nonempty). Let the sign of (T,S) be equal to $(-1)^i$; i.e., the height of T. Because the n^{th} elementary and homogeneous symmetric functions are counted by these column strict tableaux, the sum in the statement of the lemma is equal to $\sum \text{sign}(T,S)$ where the sum runs over all $(T,S) \in \mathcal{T}$. To prove Lemma 1.1, a sign-reversing weight-preserving involution I with no fixed points will be defined on $\mathcal{T}$.

Given $(T,S) \in \mathcal{T}$, let j be the integer in the bottom cell of T and k the integer in the leftmost cell of S. If $j \leq k$, let $I((T,S))$ be the element in $\mathcal{T}$ formed by removing j from T and placing it in the beginning of S. If $j > k$, let $I((T,S))$ be the element in $\mathcal{T}$ formed by removing k from S and placing it below j in T. By definition, the map I is an involution and since I changes the height of T by one cell, I is sign-reversing. There are no fixed points and because there are the same integer labels on the totality of cells, the involution is weight-preserving. An example of the involution I may be found below.

$$\begin{array}{|c|}\hline 5\\ 4\\ 2\\ 1\\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 1 & 1 & 2 & 4 & 5 \\ \hline\end{array} \quad \longleftrightarrow \quad \begin{array}{|c|}\hline 5\\ 4\\ \hline 2\\ \hline\end{array} \quad \begin{array}{|cccccc|}\hline 1 & 1 & 1 & 2 & 4 & 5 \\ \hline\end{array}$$

This completes the proof. $\square$

For $n \geq 1$, define the n^{th} power symmetric function p_n to be the sum $\sum_i x_i^n$. Given $\lambda \vdash n$, let $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$. The set $\{p_\lambda : \lambda \vdash n\}$ is a basis for Λ_n—this fact will follow later from Theorem 1.15. The generating function $H(t)$ may aid in finding a generating function involving the n^{th} power symmetric function. That is,

$$\sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \sum_{n=1}^{\infty} \frac{t^n}{n} \sum_i x_i^n = \sum_i \sum_{n=1}^{\infty} \frac{(x_i t)^n}{n}.$$

Using the series expansion for $-\ln(1-x)$, the above equation is equal to

$$\sum_i \ln \frac{1}{1 - x_i t} = \ln \prod_i \frac{1}{1 - x_i t} = \ln \left(\sum_{n=0}^{\infty} h_n t^n \right). \tag{1.5}$$

Equation (1.5) may also be proved by the exponential formula, our Theorem 3.10.

Let $ST_{(n)}$ be the set of all column strict tableaux of shape (n) where every integer in the tableau is the same. The n^{th} power symmetric function is the weighted sum over all $T \in ST_{(n)}$. With this understanding of the power symmetric functions, the following two identities in Lemma 1.2 may be proved combinatorially.

Lemma 1.2. *For $n \geq 1$,*

$$nh_n = \sum_{i=0}^{n-1} h_i p_{n-i} \qquad \textit{and} \tag{1.6}$$

$$ne_n = \sum_{i=0}^{n-1} (-1)^{n-i-1} p_{n-i} e_i. \tag{1.7}$$

Proof. The left hand side of (1.6) is counted by $R \in CS_{(n)}$ where one cell is shaded (h_n gives R and the factor of n allows for the shading). The sum on the right hand side of (1.6) is counted by pairs (S, T) where $S \in CS_{(i)}$ and $T \in ST_{(n-i)}$ for some $0 \leq i \leq n-1$. To prove (1.6), we will provide a bijection between these two collections of objects.

Suppose the shaded cell in R contains the integer j and suppose to the right of the shaded cell there are $n-i-1$ more occurrences of the integer j. Create S from R by removing the $n-i$ occurrences of j after and including the shaded cell from R and take $T \in ST_{(n-i)}$. This process is reversible, thereby providing the desired bijection. For clarity, we have displayed an example of this process:

$$\begin{array}{|ccccccccc|}\hline 1 & 1 & 2 & 3 & \boxed{3} & 3 & 3 & 4 & 5 \\ \hline\end{array} \quad \longleftrightarrow \quad \begin{array}{|cccccc|}\hline 1 & 1 & 2 & 3 & 4 & 5 \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline 3 & 3 & 3 \\ \hline\end{array}$$

Let $\mathcal{S}$ be the set of pairs (S, T) where $S \in CS_{(1^i)}$ and $T \in ST_{(n-i)}$ for some $0 \le i \le n-1$. Define the sign of $(S,T) \in \mathcal{S}$ to be $(-1)^{n-i-1}$. It follows that the sum of signs of elements in $\mathcal{S}$ is equal to the right side of (1.7). To show (1.7), a sign-reversing weight-preserving involution $\mathcal{I}$ on $\mathcal{S}$ will be provided after which point it will be shown that the fixed points correspond to a column strict tableau of shape (1^n) where one cell is shaded.

Take $(S,T) \in \mathcal{S}$. Suppose the integer j appears in T and j does not appear in S. Let $\mathcal{I}((S,T))$ denote the pair (S,T) where one integer j is removed from T and added to S. If the integer j does appear in S, let $\mathcal{I}((T,S))$ denote the pair (S,T) where the cell labeled with j in S and added to T. Then $\mathcal{I}$ is an involution which reverses sign (the length of T changes by one cell). Because the integer labels on the cells are not changed, $\mathcal{I}$ is weight-preserving. Below is an example of the involution $\mathcal{I}$.

$$\begin{array}{|c|}\hline 7\\ 6\\ 4\\ 2\\ 1\\ \hline\end{array} \quad \begin{array}{|c|c|c|c|}\hline 3&3&3&3\\ \hline\end{array} \quad \longleftrightarrow \quad \begin{array}{|c|}\hline 7\\ 6\\ 4\\ 3\\ 2\\ 1\\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 3&3&3\\ \hline\end{array}$$

The fixed points of this involution are those $(S,T) \in \mathcal{S}$ where $S \in CS_{(n-1)}$, $T \in CS_{(1)}$, and the integer in the cell in T does not appear in S. These elements have positive sign and naturally correspond to column strict tableau of shape (n) where one cell is shaded by shading the cell coming from T. This proves (1.7) and the lemma. $\square$

COROLLARY 1.3.
$$\sum_{n=1}^{\infty} p_n t^n = \frac{-\sum_{n=1}^{\infty} n e_n (-t)^n}{\sum_{n=0}^{\infty} e_n(-t)^n}.$$

PROOF. From (1.7) in Lemma 1.2,
$$\begin{aligned} E(-t)\sum_{n=1}^{\infty} p_n t^n &= \left(\sum_{i=0}^{\infty}(-1)^i e_i t^i\right)\left(\sum_{n=1}^{\infty} p_n t^n\right) \\ &= \sum_{n=1}^{\infty}\left(\sum_{i=0}^{n-1} p_{n-i}(-1)^i e_i\right) t^n \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} n e_n t^n. \end{aligned}$$
Therefore,
$$\sum_{n=1}^{\infty} p_n t^n = \frac{\sum_{n=1}^{\infty}(-1)^{n-1} n e_n t^n}{E(-t)} = \frac{-\sum_{n=1}^{\infty} n e_n(-t)^n}{\sum_{n=0}^{\infty} e_n(-t)^n}. \qquad \square$$

Let $\sigma \in S_n$ and write σ in cyclic notation such that the lengths of the cycles are written in increasing order. The cycle type of a permutation $\sigma \in S_n$ is the partition λ with part λ_i equal to the length of the i^{th} cycle. For example, the cycle type of the permutation (3)(4)(2 1 5)(6 8 7) is $(1^2, 3^2)$. Let C_λ be the set of permutations of cycle type λ

Lemma 1.4. *The number of permutations in C_λ is $n!/z_\lambda$.*

Proof. Suppose that $\lambda = (1^{m_1}, \ldots, n^{m_n})$. Naïvely parse any permutation in S_n with parentheses to create a permutation of cycle type λ. There are $n!$ ways to do this. Any one of i cyclic rearrangements of a cycle of length i leaves the permutation unchanged; divide by $1^{m_1} \cdots n^{m_n}$ to account for this. Any permutation of the m_i cycles of length i will also not change the permutation; division by $m_1! \cdots m_n!$ will resolve this. Therefore the number of permutations in C_λ is

$$|C_\lambda| = \frac{n!}{1^{m_1} \cdots n^{m_n} m_1! \cdots m_n!} = \frac{n!}{z_\lambda}. \qquad \square$$

More useful, well known identities involving the power symmetric functions are found in Lemma 1.5.

Lemma 1.5. *For $n \geq 1$,*

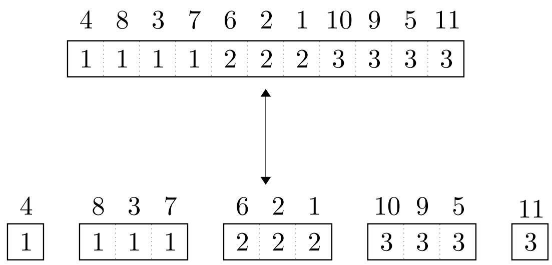
$$h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda \qquad \textit{and} \tag{1.8}$$

$$e_n = \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} p_\lambda. \tag{1.9}$$

Proof. Instead of (1.8), we will show

$$n! h_n = \sum_{\lambda \vdash n} |C_\lambda| p_\lambda \tag{1.10}$$

which by Lemma 1.4 implies the desired result. To count the left hand side of (1.10), take a $T \in CS_{(n)}$ and above it write a permutation of n. The number of ways to do this is $n!h_n$. Starting with the cell in $T \in CS_{(n)}$ with the largest integer label, say i, find the largest integer in the permutation above an i, say j. Chop the $T \in CS_{(n)}$ into two parts to create a $S \in ST_{(\ell)}$ where each cell is labeled with i and a cycle starting with the integer j. Iteratively continue this process with the remaining portion of T. We find after this process a permutation of cycle type λ along with $\ell(\lambda)$ elements in ST. This is counted by $|C_\lambda| p_\lambda$. Below we give an example of this process where $\lambda = (1^2, 3^3)$ and the corresponding element in C_λ is (4)(8 3 7)(6 2 1)(10 9 5)(11).



This process is a bijection since the inverse map may be described. Take an element $\sigma \in C_\lambda$ and elements in $ST_{(\lambda_1)}, \ldots, ST_{(\lambda_\ell)}$. Write the cycles of σ such that the lengths weakly increase when read from left to right, the maximum element within each cycle appears first, and cycles of equal length are written in decreasing order according to maximal element. To produce a permutation of n and an element in $CS_{(n)}$, write σ above the elements of $ST_{(\lambda_1)}, \ldots, ST_{(\lambda_\ell)}$. Then, place these objects in weakly increasing order first according to the repeated element in $ST_{(\lambda_1)}, \ldots, ST_{(\lambda_\ell)}$ and next according to the maximal element in the cycle above each $ST_{(\lambda_i)}$. This process will give an object like that appearing at the bottom of the figure given earlier in this proof. Glue the parts of this object together to form the desired permutation and element in $CS_{(n)}$. This describes the inverse map and explicitly verifies that our map produces a bijection.

Instead of proving (1.9), we will show

$$n!e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |C_\lambda| p_\lambda. \tag{1.11}$$

Momentarily ignoring the factor of $(-1)^{n-\ell(\lambda)}$ in the above equation, start by applying the same bijection that proved (1.10) to the right hand side of (1.11). When this is done, we find a $T \in CS_{(n)}$ below a permutation together with a factor of $(-1)^{n-\ell(\lambda)}$ (where λ is the cycle structure of a permutation formed by the method in the bijection). Let us perform an involution on these objects to rid ourselves of anything with a negative sign.

Scan the cells of $T \in CS_{(n)}$ from left to right looking for the first occurrence of two consecutive cells with the same label, say i. When this happens, find the largest two numbers in the permutation above cells labeled i in T and switch their places. This process is an involution. From the process to form the partition λ, the length of λ is changed by 1 when this involution is applied. Therefore, the involution is sign-reversing. The fixed points correspond to $T \in CS_{(n)}$ where no two consecutive cells have the same label and there is a permutation atop T. The total sign of this object is $(-1)^{n-n}$; hence these fixed points are counted by $n!e_n$. □

Although we make light use of them in this work, the most important basis in the ring of symmetric functions with respect to its relationship to other areas of mathematics is the Schur basis. Let $\lambda \vdash n$ and define

$$s_\lambda = \sum_{T \in CS_\lambda} w(T).$$

An example of one Schur symmetric function is

$$s_{(1,2)}(x_1, x_2, x_3) = 2x_1x_2x_3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2,$$

however, the most convenient way to think of these functions in terms of column strict tableaux. We have been a little bit premature in calling these objects symmetric functions as it is not obvious from the definition that these functions are elements of Λ. This is resolved in Lemma 1.6 below when we recount a well known proof of Bender and Knuth [**9**].

LEMMA 1.6. *For $\lambda \vdash n$, $s_\lambda \in \Lambda_n$.*

PROOF. Since any element in the symmetric group may be written as a product of transpositions of adjacent elements, it is enough to show that $s_\lambda(x_1, \ldots, x_N)$ is unchanged under the action of switching x_i and x_{i+1} for $i = 1, \ldots, N-1$. That is, we need to show that

$$s_\lambda(x_1, \ldots, x_i, x_{i+1}, \ldots, x_N) = s_\lambda(x_1, \ldots, x_{i+1}, x_i, \ldots, x_N).$$

for every $i = 1, \ldots, N-1$. From the definition of the Schur functions, this is equivalent to proving that for every column strict tableau of shape λ there exists a column strict tableau of shape λ with the number of occurrences of i and $i+1$ switched. We will prove this bijectively.

Take $T \in CS_\lambda$. From the definition of column strict tableaux, the appearances of i in relationship to the appearances of $i+1$ in T must be something like the appearances of the 3's and 4's below.

3	3	4	4	4	4							
			3	3	3	4	4	4				
							3	3	3	3	3	4

That is, each row in T may have a sequence of i's immediately followed by a sequence of $i+1$'s and these rows may be aligned so that no two i's or no two $i+1$'s appear on top of each other.

Let us consider a sequence s of all of the i's and $i+1$'s within a row of T such that no $i+1$'s overlap the i's in a row above and no i's overlap the $i+1$'s in a row below. For example, if we were looking at the top row in the above column strict tableau, our attention is only on the first three cells. In short, we are ignoring all the i's and $i+1$'s in T which overlap another sequence of i's and $i+1$'s in finding the sequence s.

Suppose there are j i's and k $i+1$'s in such a sequence s. Modify s so that k i's are followed by j $i+1$'s. Make this modification to every sequence s in every row of T to form a column strict tableau $\hat{T}$. Below we give this action on the T found above.

3	4	4	4	4	4							
			3	3	3	3	4	4				
							3	3	3	4	4	4

It is not difficult to see that $\hat{T}$ is a column strict tableau of shape λ, the number of i's in T is the number of $i+1$'s in $\hat{T}$, and that the number of $i+1$'s in T is the number of i's in $\hat{T}$. This correspondence is the desired bijection. $\square$

When λ is a partition of the form $(1^k, n)$ for nonnegative integers n, k, λ is called a hook shape. Schur functions corresponding to a hook shape are known as hook-Schur functions.

LEMMA 1.7. *For n, k such that $n + k \geq 1$,*

$$s_{(1^k,n)} = \sum_{i=0}^{k} (-1)^{k-i} e_i h_{n+k-i}.$$

PROOF. The involution I from the second proof of Lemma 1.1 can be applied to the right hand side. For $i < k$ in the sum, all terms in the right hand side cancel under I. Unlike Lemma 1.1 however, fixed points points remain corresponding to the case when $i = k$. The fixed points all have positive sign and are pairs (T, S) where $T \in CS_{(1^k)}$, $S \in CS_{(n)}$, and the integer in the first cell of S is smaller than the integer in the bottom cell of T. By gluing T atop S, these correspond to elements in $CS_{(1^k,n)}$—in other words, these objects count $s_{(1^k,n)}$. $\square$

COROLLARY 1.8.

$$\sum_{n=1}^{\infty} s_{(1^k,n)}t^{n+k} = \frac{-\sum_{n=k+1}^{\infty}(-1)^k e_n(-t)^n}{\sum_{n=0}^{\infty} e_n(-t)^n}.$$

PROOF. Using Lemma 1.7,

$$\begin{aligned}\sum_{n=1}^{\infty} s_{(1^k,n)}t^{n+k} &= \sum_{n=1}^{\infty}\left(\sum_{i=0}^{k}(-1)^{k-i}e_i h_{n+k-i}\right)t^{n+k}\\ &= \sum_{i=0}^{k}(-1)^{k-i}e_i t^i\left(\sum_{n=1}^{\infty} h_{n+k-i}t^{n+k-i}\right)\\ &= \sum_{i=0}^{k}(-1)^{k-i}e_i t^i\left(H(t)-\sum_{n=0}^{k-i} h_n t^n\right).\end{aligned}$$

By the fact that $H(t) = 1/E(-t)$ and using Lemma 1.1, this may be manipulated into

$$(-1)^k\left(\frac{\sum_{i=0}^{k}(-1)^i e_i t^i}{E(-t)} - \sum_{\ell=0}^{k}\left(\sum_{i=0}^{\ell}(-1)^i e_i h_{\ell-i}\right)t^\ell\right) = \frac{\sum_{i=0}^{k}(-1)^{i+k}e_i t^i}{E(-t)} + (-1)^{k+1}$$

which simplifies to the desired expression. □

This corollary gives us a generating function for hook-Schur functions in terms of the elementary symmetric functions. The similarity between will be explained shortly.

At this point we have defined five bases for the ring of symmetric functions: the monomial, elementary, homogeneous, power, and Schur symmetric functions. These bases are those which are normally described when studying the ring of symmetric functions. Before we give combinatorial descriptions of the transition matrices which transform one basis into another, we will introduce a new basis for Λ which depends on a function as a parameter. The motivation for this definition is found in the fact that it will be convenient to have an increased amount of versatility in the relationship between the homogeneous and elementary symmetric functions.

Let ν be a function on the set of nonnegative integers. Recursively define $p_{n,\nu} \in \Lambda_n$ such that

$$p_{n,\nu} = (-1)^{n-1}\nu(n)e_n + \sum_{k=1}^{n-1}(-1)^{k-1}e_k p_{n-k,\nu}$$

for all $n \geq 1$. This means that

$$\begin{aligned}E(-t)\sum_{n\geq 1} p_{n,\nu}t^n &= \left(\sum_{n\geq 0}(-1)^n e_n t^n\right)\left(\sum_{n\geq 1} p_{n,\nu}t^n\right)\\ &= \sum_{n\geq 1}\left(\sum_{k=0}^{n-1} p_{n-k,\nu}(-1)^k e_k\right)t^n\\ &= \sum_{n\geq 1}(-1)^{n-1}\nu(n)e_n t^n,\end{aligned}$$

where the last equality follows from the definition of $p_{n,\nu}$. Therefore,

$$\sum_{n\geq 1} p_{n,\nu}t^n = \frac{\sum_{n\geq 1}(-1)^{n-1}\nu(n)e_nt^n}{E(-t)} = \frac{\sum_{n\geq 1}(-1)^{n-1}\nu(n)e_nt^n}{\sum_{n\geq 0}(-1)^n e_nt^n}. \tag{1.12}$$

Define this symmetric function to be multiplicative; in other terms, for any partition λ of n, $p_{\lambda,\nu} = p_{\lambda_1,\nu}\cdots p_{\lambda_\ell,\nu}$. This basis will add a layer of versatility and adaptability to our forthcoming methods. Notice that by taking $\nu(n) = 1$ for all $n \geq 1$, we can use the above equation to show that

$$1 + \sum_{n\geq 1} p_{n,1}t^n = 1 + \frac{\sum_{n\geq 1}(-1)^{n-1}e_nt^n}{\sum_{n\geq 0}(-1)^n e_nt^n} = \frac{1}{\sum_{n\geq 0}(-1)^n e_nt^n} = 1 + \sum_{n\geq 1} h_nt^n$$

which implies $p_{n,1} = h_n$ and thus $p_{\lambda,\nu} = h_\lambda$.

Other special cases for ν give well known generating functions. Taking ν such that $\nu(n) = n$ for $n \geq 1$, $p_{n,n}$ is the power symmetric function p_n. By taking $\nu(n) = (-1)^k\chi(n \geq k+1)$ for some $k \geq 1$, $p_{n,(-1)^k\chi(n\geq k+1)}$ is the Schur function corresponding to the partition $(1^k, n)$.

Between any pair of bases for Λ_n, there is a transition matrix which writes one in terms of the other. That is, if $\{a_\lambda : \lambda \vdash n\}$ and $\{b_\lambda : \lambda \vdash n\}$ are two bases of Λ_n, let $M(a,b)_{\lambda,\mu}$ be the coefficient of a_λ in b_μ so that

$$b_\mu = \sum_{\lambda\vdash n} M(a,b)_{\lambda,\mu}a_\lambda.$$

The matrix $\|M(a,b)_{\lambda,\mu}\|_{\lambda,\mu}$ is the transition matrix from b to a. There are combinatorial interpretations for the entries of each of the transition matrices between any two standard bases for Λ_n, the majority of which were formulated by Eğecioğlu and Remmel [**7, 29**]. The proofs relating symmetric functions to permutation statistics rely heavily on them.

We have already implicitly given a combinatorial interpretation for the entries of one of these transition matrices. Let $K_{\mu,\lambda}$ be the number of column strict tableaux of shape μ with weight $x_1^{\lambda_1}\cdots x_\ell^{\ell(\lambda)}$. From the definition of the Schur symmetric functions,

$$s_\mu = \sum_{\lambda\vdash n} K_{\mu,\lambda}m_\lambda.$$

In other terms, $K_{\mu,\lambda} = M(m,s)_{\lambda,\mu}$. This matrix is known as the Kostka matrix and its entries the Kostka numbers.

THEOREM 1.9. *The set $\{s_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.*

PROOF. Since the monomial symmetric functions are a basis for Λ_n, we will show the transition matrix from the Schur symmetric functions to the monomial symmetric functions is nonsingular. This will imply that the Schur symmetric functions are also a basis for Λ_n.

The only column strict tableau of shape λ with weight $x_1^{\lambda_1}\cdots x_\ell^{\ell(\lambda)}$ is the column strict tableau where the bottom row contains all 1's, the next contains all 2's and so on. We have provided the only column strict tableau of shape $(2,2,4,6)$ with weight $x_1^6x_2^4x_3^2x_4^2$:

$$\begin{array}{|cccccc}
\cline{1-2} 4 & \multicolumn{1}{c|}{4} & & & & \\
3 & \multicolumn{1}{c|}{3} & & & & \\
\cline{3-4} 2 & 2 & 2 & \multicolumn{1}{c|}{2} & & \\
\cline{5-6} 1 & 1 & 1 & 1 & 1 & \multicolumn{1}{c|}{1} \\
\hline
\end{array}$$

Therefore, for any $\lambda \vdash n$, $K_{\lambda,\lambda} = 1$. In addition, if $\lambda \prec \mu$ in the reverse lexicographic order of partitions, there is no possible way to form a column strict tableau of shape μ with weight $x_1^{\lambda_1} \cdots x_\ell^{\ell(\lambda)}$ for similar reasons.

Index the rows and columns of the transition matrix with the reverse lexicographic order of partitions. By the above reasoning, this ordering forces $M(m, s)$ to be triangular with 1's along the diagonal; hence it is nonsingular. $\square$

Let $\mathbb{Z}_2 M_{\lambda,\mu}$ be the number of all possible $\ell(\mu) \times \ell(\lambda)$ matrices with entries either 0 or 1 such that the sum of the entries in row i gives μ_i and the sum of the entries in column j is λ_j. For example, if $\mu = (1, 2, 3)$ and $\lambda = (2, 2, 2)$, then one possible matrix with row sum μ and column sum λ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

THEOREM 1.10. *For $\mu \vdash n$,*

$$e_\mu = \sum_{\lambda \vdash n} \mathbb{Z}_2 M_{\lambda,\mu} m_\lambda.$$

PROOF. Given a $\lambda \vdash n$, let us find the number of ways we can form the monomial $x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$ by considering the terms coming from the product $e_{\mu_1} \cdots e_{\mu_j} = e_\mu$. By the definition of the monomial symmetric functions and since e_μ is a symmetric function, this will be $M(m, e)_{\lambda,\mu}$.

Consider a table where the rows are indexed by $e_{\mu_1}, \dots, e_{\mu_j}$ and the columns are indexed by $x_1, \dots, x_\ell$. Place a "1" in the i, j entry of such a table if the term x_j will come from the e_i to contribute to the monomial of type $x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$ and place a "0" in the table otherwise. An example of such a table is given when $\mu = (2^2, 3^2)$ and $\lambda = (1^2, 2, 3^2)$:

	x_1	x_2	x_3	x_4	x_5
e_2	0	0	1	0	1
e_2	0	1	0	1	0
e_3	0	0	1	1	1
e_3	1	0	0	1	1
	1	1	2	3	3

Because the elementary symmetric functions are square free, the total number of ways to create such tables where the monomial $x_1^{\lambda_1} \cdots x_\ell^{\lambda_\ell}$ is formed is the coefficient of m_λ in e_μ. This is also the number of matrices with entries either 0 or 1 with row sums μ and column sums λ. $\square$

COROLLARY 1.11. *The set $\{e_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.*

PROOF. As in the proof of Theorem 1.9, we will prove that the transition matrix from the elementary symmetric functions to the monomial symmetric functions is nonsingular.

Let $A_\lambda = \|a_{i,j}\|$ be the $\ell(\lambda) \times \max(\lambda)$ matrix such that $a_{i,j} = 0$ for $1 \le j \le \max(\lambda) - \lambda_i$ and $a_{i,j} = 1$ for $\max(\lambda) - \lambda_i + 1 \le j \le \max(\lambda)$. For example, if $\lambda = (1, 2, 4)$,

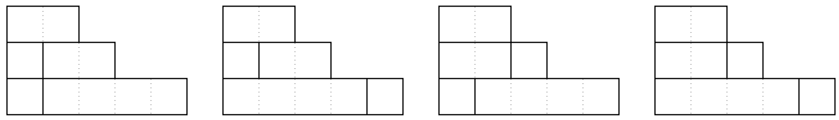
$$A_\lambda = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The mirror image of the Ferrers diagram of λ is represented in 1's in the matrix A_λ. The column sums of A_λ induce the conjugate partition λ' and A_λ is the only matrix with entries either 0 or 1 with row sum λ and column sum λ'. Therefore, $\mathbb{Z}_2 M_{\lambda,\lambda'} = 1$ for $\lambda \vdash n$. In addition, if $\mu \prec \lambda'$ in the reverse lexicographic order of partitions, there is no possible way to form a matrix with entries either 0 or 1, row sum λ, and column sum μ because the largest possible part we can form from a matrix with row sum λ is $\max(\lambda)$ as we have done in the matrix A_λ.

Normally one would consider the partitions indexing the columns and rows of a transition matrix in reverse lexicographic order; however, to show that the matrix is nonsingular, we may choose any order we wish. Write the columns and rows of $\mathbb{Z}_2 M$ in order such that if $\mu < \lambda'$, then the μ^{th} row of $\mathbb{Z}_2 M$ appears after that of λ. By the argument above, this ordering forces $\mathbb{Z}_2 M$ to be triangular with 1's along the diagonal; hence it is nonsingular. $\square$

Corollary 1.11 implies that $e_0, e_1, \ldots$ are algebraically independent and generate Λ.

Given two partitions λ and μ, let us define a object known as a brick tabloid of shape μ and type λ. The set of all such objects will be denoted by $B_{\lambda,\mu}$. A $T \in B_{\lambda,\mu}$ is formed by partitioning the rows of the Ferrers diagram of λ into "bricks" such that the lengths of the bricks induce the partition μ. For example, we now show all possible brick tabloids of shape $(2, 3, 5)$ and type $(1^2, 2^2, 4)$:



Theorem 1.12. *For $\mu \vdash n$,*

$$h_\mu = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,\mu}| e_\lambda.$$

Proof. To unclutter notation, let $M(e,h)_{\lambda,\mu} = M_{\lambda,\mu}$ be the coefficient of e_λ in h_μ for the remainder of this proof. If $\lambda \vdash n$, let $\lambda \setminus i$ be the partition λ with a part of size i removed. In the case where λ does not have a part of this size, $\lambda \setminus i$ is undefined and $M_{\lambda\setminus i,\mu} = 0$ by convention.

First, we will show that the numbers $M_{\lambda,\mu}$ satisfy the following:

(1) $M_{(n),(n)} = (-1)^{n-1}$,
(2) $M_{\lambda,(n)} = \sum_{i=1}^{n-1} (-1)^{i-1} M_{\lambda\setminus i,(n-i)}$ for λ a partition of n with more than one part, and
(3) $M_{\lambda,\mu} = \sum M_{\alpha,(\mu_1)} M_{\beta,\mu\setminus\mu_1}$ where the sum runs over all possible partitions $\alpha \vdash \mu_1$ and $\beta \vdash n - \mu_1$ such that the multiset union of the parts of α and β is equal to λ (written $\alpha + \beta = \lambda$) and μ is a partition of n with more than one part.

Lemma 1.1 may be rewritten to read

$$h_n = (-1)^{n-1}e_n + \sum_{i=1}^{n-1}(-1)^{i-1}e_i h_{n-i}. \tag{1.13}$$

The right hand side of (1.13) is equal to

$$\begin{aligned}(-1)^{n-1}e_n + \sum_{i=1}^{n-1}(-1)^{i-1}e_i \sum_{\alpha\vdash n-i} M_{\alpha,(n-i)}e_\alpha \\ = (-1)^{n-1}e_n + \sum_{\lambda\vdash n}\left(\sum_{i=1}^{n-1}(-1)^{i-1}M_{\lambda\backslash i,(n-i)}\right)e_\lambda.\end{aligned}$$

Picking the coefficient of e_n on the right hand side of the above equation, $M_{(n),(n)} = (-1)^{n-1}$. Moreover, $M_{\lambda,(n)} = \sum_{i=1}^{n-1}(-1)^{i-1}M_{\lambda\backslash i,(n-i)}$. This verifies items 1 and 2 on our list. As for item 3, consider

$$\begin{aligned}\sum_{\lambda\vdash n} M_{\lambda,\mu}e_\lambda &= h_{(\mu_1)}h_{\mu\backslash\mu_1} \\ &= \sum_{\alpha\vdash\mu_1} M_{\alpha,(\mu_1)}e_\alpha \sum_{\beta\vdash n-\mu_1} M_{\beta,\mu\backslash\mu_1}e_\beta \\ &= \sum_{\substack{\alpha\vdash\mu_1\\ \beta\vdash n-\mu_1}} M_{\alpha,(\mu_1)}M_{\beta,\mu\backslash\mu_1}e_\alpha e_\beta. \end{aligned} \tag{1.14}$$

Comparing the coefficient on both sides of (1.14) shows item 3.

The list items 1–3 completely determine the numbers $M_{\lambda,\mu}$ recursively. To complete the proof of the theorem, it remains to be shown that $(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}|$ satisfy the same three identities.

There is only one brick tabloid of shape (n) and type (n)—the brick tabloid consisting of one brick of length n inside one row of length n. Therefore, when $\lambda, \mu = (n)$, $(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}| = (-1)^{n-1}$, verifying item 1.

Item 2 is found by sorting brick tabloids of shape (n) according to the length of the first brick. Suppose $\lambda \neq (n)$ and i is a part of λ. Let $B_{\lambda,(n),i}$ be the set of $T \in B_{\lambda,(n)}$ where the first brick in T has length i. It follows that $|B_{\lambda,(n),i}| = |B_{\lambda\backslash i,(n-i)}|$. Thus,

$$\begin{aligned}(-1)^{n-\ell(\lambda)}|B_{\lambda,(n)}| &= (-1)^{n-\ell(\lambda)}\sum_{i=1}^{n-1}|B_{\lambda,(n),i}| \\ &= \sum_{i=1}^{n-1}(-1)^{i-1}\left((-1)^{(n-i)-(\ell(\lambda)-1)}|B_{\lambda\backslash i,(n-i)}|\right),\end{aligned}$$

verifying item 2.

Finally, item 3 is found by sorting brick tabloids of shape μ according to the bricks found in the top row. Suppose $B_{\lambda,\mu,\alpha}$ is the set of all $T \in B_{\lambda,\mu}$ where the first row in T has bricks which induce the partition α. It follows that $|B_{\lambda,\mu,\alpha}| = |B_{\alpha,(\mu_1)}||B_{\beta,\mu\backslash\mu_1}|$ where $\beta = \lambda - \alpha$ and therefore

$$(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}| = \sum_{\substack{\alpha\vdash\mu_1,\ \beta\vdash n-\mu_1\\ \alpha+\beta=\lambda}} (-1)^{\mu_1-\ell(\alpha)}|B_{\alpha,(\mu_1)}|(-1)^{(n-\mu_1)-\ell(\beta)}|B_{\beta,\mu\backslash\mu_1}|.$$

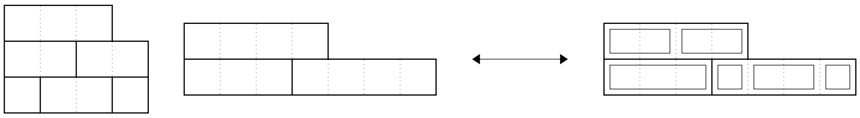
This checks item 3 and completes the proof of the theorem. □

The symmetry in the relationship between the homogeneous and elementary symmetric functions in Lemma 1.1 suggests that the λ, μ entry of the transition matrix which writes the elementary basis in terms of the homogeneous symmetric functions is also equal to $(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}|$. We now give a proof of this fact which is in the same spirit as many of our later results.

THEOREM 1.13. *The matrix* $\left\|(-1)^{n-\ell(\lambda)}|B_{\lambda,\mu}|\right\|_{\lambda,\mu}$ *is its own inverse. That is, for all* $\lambda, \mu \vdash n$,

$$\sum_{\alpha \vdash n} (-1)^{n-\ell(\lambda)+n-\ell(\alpha)}|B_{\lambda,\alpha}||B_{\alpha,\mu}| = \chi(\lambda = \mu).$$

PROOF. Fix $\lambda, \mu \vdash n$. Given $B_1 \in B_{\lambda,\alpha}$ and $B_2 \in B_{\alpha,\mu}$ for some $\alpha \vdash n$, let us form one "double brick tabloid" from B_1 and B_2 by placing the rows of B_1 in for the bricks in B_2 reading top to bottom. Call this new object $D(B_1, B_2)$. An example of this process is found below.

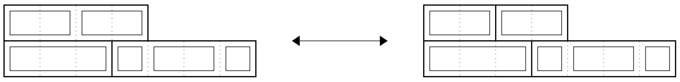


Let us call the larger of the two types of bricks "big bricks" and let us call the bricks found inside big bricks "little bricks". The sign of $D(B_1, B_2)$ is $(-1)^b$ where b is the total number of both big bricks and little bricks. Let $\mathcal{D}_{\lambda,\mu}$ be the set of all possible double brick tabloids formed in this way. It follows that the sum in the statement of this lemma is equal to $\sum \operatorname{sign}(D)$ where the sum runs over all possible $D \in \mathcal{D}_{\lambda,\mu}$. To complete the proof, we will give a sign-reversing involution on $\mathcal{D}_{\lambda,\mu}$ where precisely one fixed point of positive sign arises provided $\lambda = \mu$.

Scan the rows of $D \in \mathcal{D}_{\lambda,\mu}$ from top to bottom. For each row, read it from left to right, looking for the first occurrence of one of the following two situations:

(1) there are two consecutive big bricks, or
(2) there are two consecutive little bricks within one big brick.

If situation 1 occurs first, combine the two consecutive big bricks into one big brick. If situation 2 occurs first, break the big brick into two big bricks between two consecutive little bricks. An example of this action is below.



This process is easily seen to be a sign-reversing involution. Furthermore, the only possible fixed point is an element in $\mathcal{D}_{\lambda,\mu}$ where each row contains one little brick inside of one big brick. In this case, $\lambda = \mu$ and the sign of this element is positive. □

COROLLARY 1.14. *The set* $\{h_\lambda : \lambda \vdash n\}$ *is a basis for* Λ_n.

PROOF. According to Theorem 1.13, the matrix which writes the homogeneous symmetric functions in terms of the elementary symmetric functions (and vice-versa) is invertible. Since by Corollary 1.11 the elementary symmetric functions are a basis for Λ_n, the homogeneous symmetric functions must be a basis for Λ_n as well. □

Suppose $T \in B_{\lambda,\mu}$ has bricks of length $b_1, \ldots, b_\ell$ ending each row. Define $w_\nu(T)$ to be the product $\nu(b_1)\cdots\nu(b_\ell)$. Let $w_\nu(B_{\lambda,\mu})$ be the sum of weights of all $T \in B_{\lambda,\mu}$. For example, the brick tabloids on page 18 have weights $\nu(2)\nu(2)\nu(4)$,$\nu(2)\nu(2)\nu(1)$, $\nu(2)\nu(1)\nu(4)$, and $\nu(2)\nu(1)\nu(1)$ reading left to right. "Regular" brick tabloids is the special case found when taking $\nu(n) = 1$.

THEOREM 1.15. *For all $\mu \vdash n$,*

$$p_{\mu,\nu} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_\nu(B_{\lambda,\mu}) e_\lambda.$$

PROOF. This proof is almost identical to the proof of Theorem 1.12. Let $M(e, p_{\cdot,\nu})_{\lambda,\mu}$ be the coefficient of e_λ in $p_{n,\nu}$. The numbers $M(e, p_{\cdot,\nu})_{\lambda,\mu}$ satisfy the recursive identities

(1) $M(e, p_{\cdot,\nu})_{(n),(n)} = (-1)^{n-1}\nu(n)$,
(2) $M(e, p_{\cdot,\nu})_{\lambda,(n)} = \sum_{k=1}^{n-1} (-1)^{k-1} M(e, p_{\cdot,\nu})_{\lambda\setminus k,(n-k)}$, and
(3) $M(e, p_{\cdot,\nu})_{\lambda,\mu} = \sum M(e, p_{\cdot,\nu})_{\alpha,(\mu_1)} M(e, p_{\cdot,\nu})_{\beta,\mu\setminus\mu_1}$ where the sum runs over all possible partitions $\alpha \vdash \mu_1$ and $\beta \vdash n - \mu_1$ such that $\alpha + \beta = \lambda$.

Proofs of the fact that both $M(e, p_{\cdot,\nu})_{\lambda,\mu}$ and $(-1)^{n-\ell(\lambda)} w_\nu(B_{\lambda,\mu})$ satisfy the completely deterministic recursions above are so similar to the proof of Theorem 1.12 that they are left to the reader. □

COROLLARY 1.16. *If $\nu(n) \neq 0$ for all $n \geq 1$, the set $\{p_{\lambda,\nu} : \lambda \vdash n\}$ is a basis for Λ_n.*

PROOF. By definition of the reverse lexicographic order of partitions, if $\mu \preceq \lambda$, then there is no possible brick tabloid of shape λ and type μ because one of the bricks from μ will be too large to fit into a row of λ. Moreover, if $\lambda = \mu$, there is precisely one brick tabloid—the brick tabloid where each row contains only one brick. Therefore, when rows and columns are indexed by partitions written in the reverse lexicographic order, $M(e, p_{\cdot,\nu})_{\lambda,\mu}$ is triangular with nonzero diagonal entries. Therefore, $M(e, p_{\cdot,\nu})$ is nonsingular and since the elementary symmetric functions are a basis for Λ_n, so are the power symmetric functions. □

This concludes our brief introduction to the theory of symmetric functions. We have only included those ideas needed to develop our method of building generating functions. Those wanting an involved development of the beautiful subject and its connections to other branches of mathematics are referred to [**54, 64, 67**].

1.3. Brenti's homomorphism

We are now ready to describe the relationship between the theory of permutation statistics and symmetric functions. Understanding the proofs in this section is critical to understanding the methods described in the next chapters.

Let f_1 be a function on the nonnegative integers such that $f_1(n) = 1$ if $n = 0$ and $f_1(n) = -y(x-y)^{n-1}$ if $n \geq 1$ and define a ring homomorphism $\xi^{f_1} : \Lambda_n \to \mathbb{Q}[x, y]$ such that for $n \geq 1$,

$$\xi^{f_1}(e_n) = \frac{(-1)^n}{n!} f_1(n) \tag{1.15}$$

This definition uniquely extends to all of Λ because products of elementary symmetric functions are a basis. This homomorphism and its relationship to Theorem 1.17 below are due to Brenti; however, the proof hinges on ideas established by Beck and Remmel when they reproved the results of Brenti combinatorially [**4, 6, 13, 14**]. Our entire development of finding generating functions will come from these ideas.

Given $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let the rise statistic, $\mathrm{ris}(\sigma)$, count the number of times $\sigma_i < \sigma_{i+1}$. By convention, let $\sigma_{n+1} = n+1$ so that σ_n always registers a rise.

THEOREM 1.17. †

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} = \frac{x-y}{x - y\mathbf{e}^{t(x-y)}}.$$

PROOF. First it will be shown that

$$n!\xi^{f_1}(h_n) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)}$$

after which the statement of the theorem follows shortly. To evaluate ξ^{f_1} on $n!h_n$, write h_n in terms of the elementary symmetric functions via Theorem 1.12:

$$\begin{aligned} n!\xi^{f_1}(h_n) &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \prod_{i=1}^{\ell(\lambda)} \xi^{f_1}(e_{\lambda_i}) \\ &= n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \prod_{i=1}^{\ell(\lambda)} \frac{(-1)^{\lambda_i}}{\lambda_i!} f_1(\lambda_i) \\ &= \sum_{\lambda \vdash n} \binom{n}{\lambda} |B_{\lambda,(n)}| y^{\ell(\lambda)} (x-y)^{n-\ell(\lambda)} \end{aligned} \tag{1.16}$$

where if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$,

$$\binom{n}{\lambda} = \binom{n}{\lambda_1, \ldots, \lambda_\ell} = \frac{n!}{\lambda_1! \cdots \lambda_\ell!}$$

is the usual multinomial coefficient.

(1.16) will be interpreted as a signed, weighted sum of objects on which a sign-reversing, weight-preserving involution will be performed. The fixed points under the involution will correspond to elements in S_n with the weights on the fixed point giving the number of descents and rises in the permutation.

The sum in (1.16) selects $\lambda \vdash n$. Use the $|B_{\lambda,(n)}|$ term in (1.16) to select a brick tabloid of shape (n) filled with bricks forming the partition λ. With the multinomial coefficient, select λ_1 integers from $1, \ldots, n$ to place in a brick of length λ_1 in decreasing order, λ_2 of the remaining integers to place in a brick of length λ_2 in decreasing order, etc., so that each brick contains a decreasing sequence and each integer in $1, \ldots, n$ appears once. The $(x-y)^{n-\ell(\lambda)}$ term in the sum in (1.16)

†A boldface $\mathbf{e}$ is used to distinguish the exponential function from the elementary symmetric function.

is used to label each cell not terminating a brick with either x or $-y$. Finally, place a y in each terminal cell in a brick. The set of all such objects able to be formed in this way will be denoted $\mathcal{T}_{\xi^{f_1}}$. An example of one such $T \in \mathcal{T}_{\xi^{f_1}}$ may be found below.

x	x	$-y$	y	$-y$	y	x	y	y	$-y$	x	y
12	10	8	2	7	1	6	5	3	11	9	4

Define the weight of $T \in \mathcal{T}_{\xi^{f_1}}$, $w(T)$, to be the product of the $x, -y$, and y labels in T. The above example has weight $(-1)^3x^4y^8$. We have accounted for every term in (1.16); therefore,

$$n!\xi^{f_1}(h_n) = \sum_{T\in\mathcal{T}_{\xi^{f_1}}} w(T).$$

At this point, a sign-reversing, weight-preserving involution $I_{\xi^{f_1}}$ will be defined on $\mathcal{T}_{\xi^{f_1}}$ to leave a set of fixed points with positive sign. Let $T \in \mathcal{T}_{\xi^{f_1}}$. Scan T from left to right looking for the first of the following two occurrences:

(1) a cell labeled with $-y$, or
(2) two consecutive bricks with a decrease in the labeling between them.

If situation 1 appears first, break the brick containing the $-y$ into two bricks immediately after the violation and change the $-y$ to a y. If situation 2 appears first, combine the two consecutive bricks and change the y now in the middle of the brick to a $-y$. This process is the involution $I_{\xi^{f_1}}$—it does not alter any cells labeled with x but does flip the sign on T. The image of the object found earlier in this proof under $I_{\xi^{f_1}}$ is displayed below.

x	x	y	y	$-y$	y	x	y	y	$-y$	x	y
12	10	8	2	7	1	6	5	3	11	9	4

Let $\mathcal{F}_{\xi^{f_1}}$ be the set of fixed points under the involution $I_{\xi^{f_1}}$ consisting of those $T \in \mathcal{T}_{\xi^{f_1}}$ where there are no $-y$'s and there are no decreases between two bricks. An example of $T \in \mathcal{F}_{\xi^{f_1}}$ may be found below.

x	x	x	y	x	y	x	y	y	x	x	y
12	10	8	2	7	1	6	3	5	11	9	4

The row of integers on a fixed point can be read as an element of the symmetric group S_n written in one line notation. When this is done, there is an x label above an integer if and only if that integer registers a descent and a y label above an integer if and only if that integer registers a rise. The above fixed point corresponds to 12 10 8 2 7 1 6 3 5 11 9 4 $\in S_{12}$ with seven descents and five rises. The involution $I_{\xi^{f_1}}$ implies

$$n!\xi^{f_1}(h_n) = \sum_{T\in\mathcal{T}_{\xi^{f_1}}} w(T) = \sum_{T\in\mathcal{F}_{\xi^{f_1}}} w(T) = \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)}.$$

We have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)} &= \sum_{n=0}^{\infty}t^n\xi^{f_1}(h_n)\\ &= \xi^{f_1}\left(\sum_{n=0}^{\infty}h_nt^n\right)\\ &= \xi^{f_1}\left(\sum_{n=0}^{\infty}e_n(-t)^n\right)^{-1}\end{aligned}$$

where the last equality comes from (1.4). Continuing this string of equalities, we have

$$\frac{1}{1+\sum_{n=1}^{\infty}(-t)^n(-1)^n\frac{-y(x-y)^{n-1}}{n!}} = \frac{x-y}{x-y-y\sum_{n=1}^{\infty}\frac{t^n(x-y)^n}{n!}} = \frac{x-y}{x-y\mathbf{e}^{t(x-y)}}. \quad \square$$

Because $\mathrm{des}(\sigma)+\mathrm{ris}(\sigma)=n$ for all $\sigma\in S_n$, Theorem 1.17 may be stated in terms of the indeterminate x instead of both x and y without any loss of information. We are carrying along the "extra" indeterminate y because later it will be convenient to have every fixed point have weight a monomial of degree n. Further, in later applications of the techniques we are developing, interesting asymmetries between x and y can be found.

When the homomorphism ξ^{f_1} is applied to h_μ for an arbitrary partition μ, another permutation statistic is found. Take $\sigma\in S_n$ and $\mu\vdash n$. Let $\mathrm{des}_\mu(\sigma)$ be the number of descents of σ where any descents occurring at places $\mu_1,\mu_1+\mu_2,\ldots,\mu_1+\cdots+\mu_\ell$ are ignored. For example, if $\sigma = 3\ 1\ 5\ 4\ 2\ 7\ 6\in S_7$, then $\mathrm{des}_{(2,2,3)}(\sigma)=3$ (the descent from the 4 to the 2 is not tallied). Note that $\mathrm{des}_{(n)}(\sigma)=\mathrm{des}(\sigma)$. Let $\mathrm{ris}_\mu(\sigma)$ be the statistic such that $\mathrm{des}_\mu(\sigma)+\mathrm{ris}_\mu(\sigma)=n$ for all $\sigma\in S_n$.

THEOREM 1.18. *For $\mu\vdash n$,*

$$n!\xi^{f_1}(h_\mu)=\sum_{\sigma\in S_n}x^{\mathrm{des}_\mu(\sigma)}y^{\mathrm{ris}_\mu(\sigma)}.$$

PROOF. The same steps which gave rise to (1.16) also show that

$$n!\xi^{f_1}(h_\mu)=\sum_{\lambda\vdash n}\binom{n}{\lambda}|B_{\lambda,\mu}|y^{\ell(\lambda)}(x-y)^{n-\ell(\lambda)}. \tag{1.17}$$

The sum in (1.17) selects $\lambda\vdash n$. Use the $|B_{\lambda,\mu}|$ term in (1.17) to select a brick tabloid of shape μ filled with bricks forming the partition λ. With the multinomial coefficient, fill each cell with a distinct integer from $1,\ldots,n$ such that the integers in each brick are in decreasing order. The $(x-y)^{n-\ell(\lambda)}$ term in the sum in (1.17) is used to label each cell not terminating a brick with either x or $-y$. Finally, place a y in each cell at the end of a brick. The set of objects able to be formed in this way will be called $\mathcal{T}^{\mu}_{\xi^{f_1}}$. An example of one such $T\in\mathcal{T}^{\mu}_{\xi^{f_1}}$ can be found below.

x 10	y 4			
x 12	$-y$ 5	y 1	x 9	y 3
y 6	y 2	$-y$ 11	x 8	y 7

Just as in the proof of Theorem 1.17, define the weight of $T \in \mathcal{T}^{\mu}_{\xi^{f_1}}$, $w(T)$, to be the product of the $x, -y$, and y labels in T. The above example has weight $(-1)^2 x^4 y^8$. It follows that the sum in (1.17) is equal to $\sum w(T)$ where the sum runs over all $T \in \mathcal{T}^{\mu}_{\xi^{f_1}}$.

Scan the rows of $T \in \mathcal{T}^{\mu}_{\xi^{f_1}}$ from left to right then top to bottom, looking for the first of the following two occurrences:

(1) a cell labeled with $-y$, or
(2) two consecutive bricks within a row with a decrease in the integer labeling between them.

Depending on which situation occurs first, apply the involution $I_{\xi^{f_1}}$ in the proof of Theorem 1.17 to T. That is, according to the situation, break or combine bricks accordingly. The remaining fixed points correspond to $\sigma \in S_n$ when the integers are read from left to right then top to bottom. The weight of a fixed point is equal to $x^{\mathrm{des}_{\mu}(\sigma)} y^{\mathrm{ris}_{\mu}(\sigma)}$. This completes the proof. $\square$

In general, suppose we have some statistic, say "stat", defined for $\sigma \in S_n$. It may be possible to define $\mathrm{stat}_{\mu}(\sigma)$ to be equal to $\mathrm{stat}(\sigma)$ except that any normally registering occurrence of $\mathrm{stat}(\sigma)$ at integers in the places $\mu_1, \mu_1+\mu_2, \ldots, \mu_1+\cdots+\mu_\ell$ in σ are ignored. In the rest of this document we will prove that many statistics arise from applying assorted homomorphisms to h_n. After this is done, in many cases, we may form a μ version of this statistic and prove a result close to that in Theorem 1.18. The method of proving such a result follows the ideas in Theorem 1.18. To avoid redundancy and to reserve space for the most spectacular generating functions and permutation statistics, we will not mention this fact throughout the rest of this document; however, whenever we prove a theorem involving applying a homomorphism to h_n there may be an implicit corollary concerning the corresponding μ statistic.

Another generalization of Theorem 1.17 may be found by applying the basic homomorphism ξ^{f_1} to hook-Schur symmetric functions. Fix $k \geq 0$ and let $S_n^{\downarrow_k}$ be the subset of S_n where every element ends with at least $k-1$ descents. That is, for $\sigma \in S_n$, it must be the case that $\sigma_{n-k} > \sigma_{n-k+1} > \cdots > \sigma_n$. For example, $9\ 12\ 11\ 5\ 6\ 7\ 10\ 2\ 8\ 4\ 3\ 1 \in S_{12}^{\downarrow_4}$.

THEOREM 1.19. *For $k \geq 0$,*

$$\sum_{n=1}^{\infty} \frac{t^{n+k}}{(n+k)!} \sum_{\sigma \in S_{n+k}^{\downarrow_k}} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} = \frac{yx^{k-1} \sum_{n=k+1}^{\infty} (t^n (x-y)^{n-k+1})/n!}{x - y\mathbf{e}^{t(x-y)}}.$$

PROOF. First it will be shown that for n, k such that $n+k \geq 1$,

$$(n+k)!\xi^{f_1}(s_{(1^k,n)}) = \frac{(-1)^k(x-y)^{k-1}}{x^{k-1}} \sum_{\sigma \in S_{n+k}^{\downarrow k}} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)}.$$

According to Lemma 1.7 and Theorem 1.12,

$$\begin{aligned}
(-1)^k s_{(1^k,n)} &= (-1)^k \sum_{i=0}^{k} (-1)^{k-i} e_i h_{n+k-i} \\
&= (-1)^k \sum_{i=0}^{k} (-1)^{k-i} e_i \sum_{\lambda \vdash n+k-i} (-1)^{n+k-i-\ell(\lambda)} |B_{\lambda,(n+k-i)}| e_\lambda \\
&= \sum_{\lambda \vdash n+k} (-1)^{n+k} |B_{\lambda,(n+k)}| e_\lambda - \sum_{i=1}^{k} (-1)^{i-1} e_i \\
&\quad \times \sum_{\alpha \vdash n+k-i} (-1)^{n+k-i-\ell(\alpha)} |B_{\alpha,(n+k-i)}| e_\alpha.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&(-1)^k (n+k)! \xi^{f_1}(s_{(1^k,n)}) \\
&\quad = (n+k)! \left(\sum_{\lambda \vdash n+k} (-1)^{n+k} |B_{\lambda,(n+k)}| \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i} \frac{-y(x-y)^{\lambda_i - 1}}{\lambda_i!} \right. \\
&\qquad - \sum_{i=1}^{k} (-1)^{-1} \frac{-y(x-y)^{i-1}}{i!} \\
&\qquad \left. \times \sum_{\alpha \vdash n+k-i} (-1)^{n+k-i-\ell(\alpha)} |B_{\alpha,(n+k-i)}| \prod_{i=1}^{\ell(\alpha)} (-1)^{\alpha_i} \frac{-y(x-y)^{\alpha_i - 1}}{\alpha_i!} \right) \\
&\quad = \sum_{\lambda \vdash n+k} \binom{n}{\lambda} |B_{\lambda,(n+k)}| y^{\ell(\lambda)} (x-y)^{n-\ell(\lambda)} \\
&\qquad - \sum_{i=1}^{k} \sum_{\alpha \vdash n+k-i} \binom{n}{i, \alpha} |B_{\alpha,(n+k-i)}| y^{\ell(\alpha)} (x-y)^{n+k-\ell(\alpha)-1}.
\end{aligned} \tag{1.18}$$

The factor of $(-1)^k$ in the above expression will be used when summing over all possible fixed points—let us ignore it for the moment. The first of the two sums in (1.18) can be used to form elements in $\mathcal{T}_{\xi^{f_1}}$ where $\mathcal{T}_{\xi^{f_1}}$ is the same set found in the proof of Theorem 1.17 (and where the total number of cells is $n+k$). The second of the two sums can produce elements in $\mathcal{T}_{\xi^{f_1}}$ where the last brick must have a length i between 1 and k (and where the total number of cells is $n+k$). Because we are taking the difference between the two sums, we are left with the subset of $\mathcal{T}_{\xi^{f_1}}$ containing those objects where the last brick must have length larger than k cells. The involution $I_{\xi^{f_1}}$ may be applied to this subset of $\mathcal{T}_{\xi^{f_1}}$ provided that we simply do not apply the involution after the k^{th} cell from the right. That is, do not split a brick into two bricks at any $-y$ after the k^{th} cell reading right to left—if this were done, we would not have an object in the subset of $\mathcal{T}_{\xi^{f_1}}$ we are considering.

Under this modified involution, we are left with fixed points like that found below.

x	x	x	y	x	y	y	x	x	$-y$	x	y
11	9	8	6	7	1	3	12	10	5	4	2

The contributory weight of the set of fixed points for a given permutation before the last k cells is $x^{\mathrm{des}(\sigma)-(k-1)}y^{\mathrm{ris}(\sigma)-1}$ and the contributory weight from the last k cells is $y(x-y)^{k-1}$. Summing over all possible fixed points and taking into account the $(-1)^k$ in (1.18), we obtain the desired expression displayed in the beginning of this proof. We now have that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{t^{n+k}}{(n+k)!}\sum_{\sigma\in S_{n+k}^{\downarrow k}} x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)} &\\ &= \xi^{f_1}\left(\frac{(-1)^k x^{k-1}}{(x-y)^{k-1}}\sum_{n=1}^{\infty}t^{n+k}s_{(1^k,n)}\right)\\ &= \frac{(-1)^k x^{k-1}}{(x-y)^{k-1}}\xi^{f_1}\left(\frac{-\sum_{n=k+1}^{\infty}(-1)^k e_n(-t)^n}{\sum_{n=0}^{\infty}e_n(-t)^n}\right)\\ &= \frac{x^{k-1}}{(x-y)^{k-1}}\left(\frac{-\sum_{n=k+1}^{\infty}(-t)^n(-1)^n\frac{-y(x-y)^{n-1}}{n!}}{\sum_{n=0}^{\infty}(-t)^n(-1)^n\frac{-y(x-y)^{n-1}}{n!}}\right),\end{aligned}$$

which by multiplying the fraction in the parenthesis by $(x-y)/(x-y)$ may be arranged to look like the statement of the theorem. □

In the case of $k=0$, Theorem 1.19 simplifies to the generating function registering descents over the symmetric group in Theorem 1.17 as it should (except one series starts at $n=0$ and the other at $n=1$ so slight modifications are needed to make them appear exactly the same).

Applying ξ^{f_1} to the n^{th} power symmetric function can give information about permutations in $C_{(n)}$, the set of permutations which are one cycle when written in cyclic notation. Let $\mathrm{fxd}(\sigma)$ be the number of fixed points in $\sigma\in S_n$.

THEOREM 1.20. † *For $n\geq 1$,*

$$(n-1)!\xi^{f_1}(p_n)=\sum_{\sigma\in C_{(n)}}x^{\mathrm{exc}(\sigma)}y^{\mathrm{exc}(\sigma^{-1})+\mathrm{fxd}(\sigma)}.$$

PROOF. Expanding p_n in terms of the elementary symmetric functions by Theorem 1.15,

$$\begin{aligned}(n-1)!\xi^{f_1}(p_n) &= (n-1)!\xi^{f_1}\left(\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}w_n(B_{\lambda,(n)})e_\lambda\right)\\ &= (n-1)!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}w_n(B_{\lambda,(n)})\prod_{i=1}^{\ell(\lambda)}(-1)^{\lambda_i}\frac{-y(x-y)^{\lambda_i-1}}{\lambda_i!}.\end{aligned}$$

†For $\sigma\in S_n$, $\mathrm{exc}(\sigma)+\mathrm{fxd}(\sigma)$ is sometimes called the number of weak excedances of σ.

It is desirable to sort the tabloids appearing in the above equation according to the lengths of the bricks due to the definition of weighted brick tabloids. The above equation is equal to

$$(1.19)\quad (n-1)!\sum_{\lambda\vdash n}\sum_{\substack{T\in B_{\lambda,(n)}\\ T\text{ has bricks}\\ \text{of length } b_1,\ldots,b_\ell}} b_\ell\frac{y^\ell(x-y)^{n-\ell}}{b_1!\cdots b_\ell!}$$
$$=\sum_{\lambda\vdash n}\sum_{\substack{T\in B_{\lambda,(n)}\\ T\text{ has bricks}\\ \text{of length } b_1,\ldots,b_\ell}}\binom{n-1}{b_1,\ldots,b_{\ell-1},b_\ell-1}y^\ell(x-y)^{n-\ell}.$$

The right hand side of (1.19) will be used to create combinatorial objects based on brick tabloids. Then, as in the previous three theorems, an involution will be applied to leave a set of fixed points corresponding to permutations with the appropriate weights.

Use the sums in the right hand side of (1.19) to select a brick tabloid of shape (n) and type λ for some λ where the bricks have lengths $b_1,\ldots,b_\ell$ when reading left to right. Use the multinomial coefficient to fill the first $n-1$ cells of the tabloid with a permutation of $2,\ldots,n$ such that integers decrease within each brick. Place the integer 1 in the last cell of the brick tabloid. Then, use the $y^\ell(x-y)^{n-\ell}$ term to label the bricks as usual. Below we give an example of such an object.

x	y	x	$-y$	y	$-y$	x	x	x	y	$-y$	y
10	5	9	6	2	12	11	8	7	3	4	1

The set of objects we are able to form in this way are the same objects as in the proof of Theorem 1.17 with the exception that the permutation written in the bottom of a tabloid must end in 1.

Let us apply the involution $I_{\xi f_1}$ to this collection of objects. The fixed points may be interpreted as a permutation of n of cycle type (n) written in a way such that the 1 appears last. For example, the corresponding permutation in the above figure is (10 5 9 6 2 12 11 8 7 3 4 1). When written in this way, the powers of x record excedances while the powers of y record excedances of the inverse permutation. In the case where $n=1$, there is one power of y to record the fixed point. Summing over all fixed points under the involution $I_{\xi f_1}$ completes the proof. $\square$

Corollary 1.21.

$$\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in C_n}x^{\mathrm{exc}(\sigma)}y^{\mathrm{exc}(\sigma^{-1})+\mathrm{fxd}(\sigma)}=\ln\left(\frac{x-y}{x-y\mathbf{e}^{t(x-y)}}\right).$$

Proof. By (1.5) and Theorem 1.17,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in C_n}x^{\mathrm{exc}(\sigma)}y^{\mathrm{exc}(\sigma^{-1})+\mathrm{fxd}(\sigma)} &= \xi^{f_1}\left(\sum_{n=1}^{\infty}\frac{t^n}{n}p_n\right)\\ &= \xi^{f_1}\left(\ln\sum_{n=0}^{\infty}h_n t^n\right)\\ &= \ln\left(\frac{x-y}{x-y\mathbf{e}^{t(x-y)}}\right)\end{aligned}$$

which completes the proof. ☐

Theorem 1.17 gives us one way to interpret the application of ξ^{f_1} on $n!h_n$. Corollary 1.22 below provides a second interpretation.

Corollary 1.22.

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\mathrm{exc}(\sigma)}y^{\mathrm{exc}(\sigma^{-1})+\mathrm{fxd}(\sigma)} = \frac{x-y}{x-y\mathbf{e}^{t(x-y)}}.$$

Proof. By the proof of Theorem 1.17, we only need to show that

$$n!\xi^{f_1}(h_n) = \sum_{\sigma\in S_n}x^{\mathrm{exc}(\sigma)}y^{\mathrm{exc}(\sigma^{-1})+\mathrm{fxd}(\sigma)}.$$

By (1.10), we have

$$\begin{aligned}n!h_n &= \sum_{\substack{\lambda\vdash n \text{ has } m_i\\ \text{parts of size } i}}\frac{n!}{m_1!\cdots m_n!\lambda_1\cdots\lambda_\ell}p_\lambda\\ &= \sum_{\substack{\lambda\vdash n \text{ has } m_i\\ \text{parts of size } i}}\frac{1}{m_1!\cdots m_n!}\binom{n}{\lambda}(\lambda_1-1)!p_{\lambda_1}\cdots(\lambda_\ell-1)!p_{\lambda_\ell}.\end{aligned}$$

Therefore, by Theorem 1.20, $n!\xi^{f_1}(h_n)$ is equal to

$$\begin{aligned}&\sum_{\substack{\lambda\vdash n \text{ has } m_i\\ \text{parts of size } i}}\frac{1}{m_1!\cdots m_n!}\binom{n}{\lambda}\\ &\times\left(\sum_{\sigma\in C_{\lambda_1}}x^{\mathrm{exc}(\sigma)}y^{\mathrm{exc}(\sigma^{-1})+\mathrm{fxd}(\sigma)}\right)\cdots\left(\sum_{\sigma\in C_{\lambda_\ell}}x^{\mathrm{exc}(\sigma)}y^{\mathrm{exc}(\sigma^{-1})+\mathrm{fxd}(\sigma)}\right).\end{aligned}\tag{1.20}$$

Use the sum in the above equation to select a cycle type for a permutation in S_n. The multinomial coefficient tells us which integers to be placed in each cycle to form a permutation of n. The product of sums coming from Theorem 1.20 allows us to create a permutation of S_n in cyclic notation and record powers of x and y according to the needed statistics. Finally, the factor of $1/m_1!\cdots m_n!$ in (1.20) will uniquely order the cycles of the permutation written in cyclic notation. We have used every term in (1.20) to create permutations in S_n while registering the desired statistics in powers of x and y. This completes the proof. ☐

Corollary 1.22 implies that descents and excedances are equidistributed over the symmetric group. Succumbing to innate combinatorial tendencies, we now prove this bijectively. Each $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ with i descents will be paired with $\tilde{\sigma} \in S_n$ with i excedances. When written in one line notation, suppose $\sigma_j = 1$. Erase the first j integers in σ and start to construct $\tilde{\sigma}$ with the cycle $(\sigma_j \ \cdots \ \sigma_2 \ \sigma_1)$. Iteratively continue this process with the next smallest integer in σ; building $\tilde{\sigma}$ cycle by cycle. For example, if $\sigma = 9\ 3\ 1\ 6\ 8\ 2\ 7\ 4\ 5$, $\tilde{\sigma}$ would be (in cyclic notation) $(1\ 3\ 9)(2\ 8\ 6)(4\ 7)(5)$. This process is a bijection. This construction only breaks the partition σ at a rise and writes the cycles in $\tilde{\sigma}$ such that a way that if $\sigma_j > \sigma_{j+1}$ for any $1 \leq j \leq n$, then $j < \tilde{\sigma}_j$. Therefore, $\mathrm{des}(\sigma) = \mathrm{exc}(\tilde{\sigma})$ and we have displayed the desired pairing. Although we will not explicitly mention it throughout this work, we note that this bijection can transform many of our future results involving the descent statistic and rewrite them in terms of excedances.

1.4. Published uses of brick tabloids in permutation enumeration

Although there have been other connections from the ring of symmetric functions to the permutation enumeration of the symmetric group, Brenti established a direct connection with the homomorphism ξ^{f_1} [**13, 14**]. As mentioned in Section 1.3, Beck and Remmel provided the ideas which we used to prove the theorems in [**4, 6**]. In addition to these works, there have been other publications which further investigate the methods introduced in the previous section (as is the case with this monograph, all authors of these publications have had direct ties to Remmel). In this section we will recount all of these uses of brick tabloids which have appeared in the literature. This will serve to clarify the progress and advancements we will display in the next few chapters.

There have been a string of publications which have included this idea of using brick tabloids to connect symmetric functions to permutation statistics. These are

- Beck's thesis together with two follow-up papers, one of which was the paper co-authored with Remmel described in detail in the previous section [**4, 5, 6**],
- a paper by Ram, Remmel, and Whitehead [**58**],
- Wagner's thesis together with a follow-up paper [**70, 71**],
- Langley's thesis together with a follow-up paper, [**51, 52**], and
- a paper by Langley and Remmel [**53**].

We will systematically review the content of each of these works. In doing so, we will describe how this monograph relates to the results in these publications while describing a unifying theory to construct a menagerie of generating functions.

Although we revised and recounted much of [**6**] in Section 1.3, we have not mentioned the extensions to q-analogues given by Beck and Remmel. In particular, they further refined Theorem 1.17 to find an expression for

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{inv}(\sigma)}$$

where $[n]_q = q^0 + \cdots + q^{n-1}$ and $[n]_q! = [n]_q \cdots [2]_q [1]_q$. We will use a similar methodology in the next section to vastly generalize the generating functions we can refine by the inversion statistic. Therefore, because of the similarity to our Section 1.5, we will not explain the techniques involved in adding this extra indeterminate found in [**6**] and parts of [**4**] at this time.

The final chapters of Beck's thesis [**4**] and in her follow-up paper [**5**], an account of the permutation enumeration of the hyperoctahedral group is given. This is a special case of a wreath product group which is defined as follows. Let G be a finite group. The group $G \wr S_n$ is defined as

$$G \wr S_n = \{(f, \sigma) \mid f : \{1, \dots, n\} \to G \text{ and } \sigma \in S_n\}$$

and is referred to as the wreath product of G with S_n. A convenient way to think of the group is by considering the set of $n \times n$ permutation matrices where each 1 is replaced with an element of G. In this light, group multiplication is defined to be matrix multiplication. Elements in $G \wr S_n$ can be presented in cyclic, one line, or matrix notation where each integer in a cycle in S_n is paired with an element in G. For example, if $g_1, \dots, g_5$ are in G, $\sigma \in G \wr S_n$ may be given by

$$\sigma = \begin{pmatrix} 0 & g_1 & 0 & 0 & 0 \\ g_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_3 & 0 \\ 0 & 0 & g_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_5 \end{pmatrix}$$

in one line notation,

$$\sigma = (g_2, 2)\ (g_1, 1)\ (g_4, 4)\ (g_3, 3)\ (g_5, 5)$$

or in cycle notation,

$$\sigma = \big((g_1, 1), (g_2, 2)\big)\big((g_3, 3), (g_4, 4)\big)\big((g_5, 5)\big).$$

Important examples of wreath products are found when G is cyclic. The group $\mathbb{Z}_2 \wr S_n$ is a Weyl group of type B and is referred to as B_n, Young's hyperoctahedral group, or the group of signed permutations. The subgroup of index 2 of B_n containing elements with an even total number of negative signs is known as D_n. This is the Weyl group of type D. The Weyl groups of type B and D appear in the study of Lie algebras and root systems. In the past decade, a number of papers have been published on the permutation statistics of B_n and, more recently, D_n [**27, 37, 38, 59, 60, 61**].

The techniques Beck found to investigate the permutation enumeration of the hyperoctahedral group involved so-called λ-ring notation which extends the ring of symmetric functions. Therefore, to understand Beck's approach and the later works containing brick tabloids, we need to introduce λ-ring notation. It is important to note, however, that none of the mathematics in the rest of this monograph hinge on the use of λ-ring notation. In our building of generating functions, we employ a more elementary approach without sacrificing power or flexibility. Thus, those readers not interested in delving into these topics can skim the rest of this section without fear of missing a crucial component in the development of building generating functions.

Let A be a set of formal commuting variables and A^* the set of words in A. The empty word will be identified with "1". Let $c \in \mathbb{C}$, $x = a_1 a_2 \cdots a_i$ be any word in A^*, and $X, X_1, X_2, \dots$ be any sequence of formal sums of the words in A^* with complex coefficients. Define λ-ring notation on the power symmetric functions by

$$p_n[0] = 0, \qquad p_n[1] = 1,$$
$$p_n[x] = x^n = a_1^n a_2^n \cdots a_i^n, \qquad p_n[c\,X] = c\,p_n[X],$$
$$p_n\left[\sum_i X_i\right] = \sum_i p_n[X_i], \qquad p_\lambda[X] = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[X],$$

where n is a positive integer and $\lambda \vdash n$. These definitions imply that $p_n[XX_1] = p_n[X]p_n[X_1]$ and therefore $p_\lambda[XX_1] = p_\lambda[X]p_\lambda[X_1]$. These definitions also imply that for any complex number c, $p_\lambda[cX] = c^{\ell(\lambda)}p_\lambda[X]$. When $X = x_1 + x_2 + \dots$, then

$$p_n[X] = \sum_{i=1} x_i^n$$

is the power symmetric function p_n and therefore $p_\lambda[X] = p_\lambda$ for any partition λ.

The power symmetric functions are a basis in the space of symmetric functions, so if Q is a symmetric function, then there are unique coefficients a_λ such that $Q = \sum_\lambda a_\lambda p_\lambda$. Define $Q[X] = \sum_\lambda a_\lambda p_\lambda[X]$. It follows that in the special case where $X = x_1 + x_2 + \dots$ is a sum of letters of A, $Q[X]$ is simply the symmetric function Q. In particular, our definition of λ-ring notation extends to the homogeneous and elementary symmetric functions. Using Lemma 1.5,

$$h_n[X] = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p_\lambda[X] \qquad \text{and}$$
$$e_n[X] = \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} p_\lambda[X].$$

The representation theory of wreath product groups can be investigated in a manner which parallels the representation theory for the symmetric group [**56**]. When viewing the representation theory for the hyperoctahedral group in this light, it becomes natural to evaluate the λ-ring version of symmetric functions at $X+Y$ and $X-Y$ for two formal sums X and Y. Therefore, when attempting to prove results about the permutation enumeration of the hyperoctahedral group, Beck modified Brenti's homomorphism to define ξ_B on the λ-ring notation version of the elementary symmetric functions by

$$\xi_B(e_n[X+Y]) = \frac{(1-x)^{n-1} + x(x-1)^{n-1}}{2^n n!}$$

and

$$\xi_B(e_n[X-Y]) = \frac{(1-x)^{n-1} - x(1-x)^{n-1}}{2^n n!}.$$

She then went on to prove that applying ξ_B to the λ-ring notation version of the homogeneous symmetric functions $h_n[X+Y]$ gave information about the permutation enumeration for the hyperoctahedral group B_n. In particular,

$$2^n n! \xi_B(h_n[X+Y]) = \sum_{\sigma \in B_n} x^{\mathrm{des}_B(\sigma)} \tag{1.21}$$

where $\mathrm{des}_B(\sigma)$ is a statistic defined on the hyperoctahedral group (which we will describe in Section 3.1).

A key component in the work of Beck is the application of these homomorphisms on $p_\lambda[X]$. She showed an analogous statement to our Theorem 1.20 for the hyperoctahedral group B_n. It was shown that ξ_B on p_λ refined conjugacy classes of B_n by an excedance-type statistic.

In [**70, 71**], Wagner continued the work of Beck when she considered the permutation enumeration of groups of the form $\mathbb{Z}_3 \wr S_n$. The representation theory of $\mathbb{Z}_3 \wr S_n$ naturally leads to evaluating λ-ring symmetric functions at $X+Y+Z$, $X+\alpha Y+\alpha^2 Z$, and $X+\alpha^2 Y+\alpha Z$ where α is a primitive third root of unity. Thus, Wagner defined a homomorphism ξ_W such that

$$\xi_W(e_n[X+Y+Z]) = \frac{(1-x)^{n-1}}{3^{n-1}n!},$$

and $\xi_W(e_n[X+\alpha Y+\alpha^2 Z])$ and $\xi_W(e_n[X+\alpha^2 Y+\alpha Z])$ are both equal to

$$(1+\alpha^n+\alpha^{2n})\frac{(1-x)^{n-1}}{3^n n!}.$$

Then it was proved that

$$3^n n!\xi_W(h_n[X+Y+Z]) = \sum_{\sigma\in\mathbb{Z}_3\wr S_n} x^{\mathrm{des}_W(\sigma)} \tag{1.22}$$

where $\mathrm{des}_W(\sigma)$ counts occurrences of a statistic similar to descents defined for elements in $\mathbb{Z}_3 \wr S_n$ after which it was indicated how to easily extend these type of results for wreath product groups of the form $\mathbb{Z}_k \wr S_n$. Wagner, like Beck, spent significant effort on the application of her homomorphisms to the power symmetric functions to learn about an excedance type statistic over given conjugacy classes. In her work it is evident that determining the effect of the application of homomorphisms on homogeneous symmetric functions is easier than that of the power symmetric functions.

The works of Beck and Wagner, which define homomorphisms on the λ-ring versions of the elementary symmetric functions, proved the results found in (1.21) and (1.22) in essentially the same way. They both used modified brick tabloids to expand the λ-ring version of the homogeneous symmetric functions in terms of the λ-ring version of the elementary symmetric functions. Although not exhibited this way, these variants on brick tabloids may be thought of as our brick tabloids introduced in Section 1.2 with certain weights attached to some of the bricks. To prove (1.21) and (1.22), these modified brick tabloids were filled with objects corresponding to elements in wreath product groups. Involutions were performed to find fixed points which were interpreted as elements in wreath product groups.

The results of Beck and Wagner will be given, together with a number of generalizations, in our Section 3.1. We will not need λ-ring notation nor will we need to be led to our results from representation theory. Instead, we will show how to appropriately modify the weighting of brick tabloids to produce the same results.

The main goal in [**58**], the paper by Ram, Remmel, and Whitehead, was to understand the properties of a symmetric function further refined by indeterminates q and t. In this paper, they described the transition matrices between this new symmetric function and the homogeneous, elementary, monomial, and Schur symmetric functions. The transition matrices had similar combinatorial interpretations as we found in Section 1.2 with the addition of extra indeterminates.

In the last section of [**58**], Brenti's homomorphism is applied to the new symmetric functions indexed by q. The q-analogue of brick tabloids developed to explain the relationship between the new q-indexed symmetric functions and the elementary symmetric functions was filled with integers. Then, a brick breaking and combining involution was applied to leave fixed points corresponding to permutations with powers of x registering excedances and powers of q counting a modified version of the inversion statistic for cycles in a permutation. This section in the paper showed that the application of Brenti's homomorphism on this new basis in the ring of symmetric functions gave a sensible result. The techniques we will give in Section 1.5 for dealing with q-analogues combined with the modified bases $p_{n,\nu}$ for symmetric functions we described in Section 1.2 will provide the same results.

The main results in [**51**] deal with the decomposition of the plethysm of special cases of Schur functions into sums of Schur functions. Toward the end of his thesis, however, Langley shows how to add a power of q to the λ-ring version of the power symmetric functions when applying Brenti, Beck, and Wagner's homomorphisms. Due to the fact that $p_\lambda[qX] = q^{\ell(\lambda)} p_\lambda[X]$, he was able to record the number of cycles in a permutation along with the number of excedances when applying these homomorphisms. Furthermore, he was able to use a power of q to count the signs of cycles in the hyperoctahedral group and $\mathbb{Z}_3 \wr S_n$ (here, the sign of the cycle is the product of the group elements attached to integers in the cycle).

Unlike the case with the other previously published works detailing the use of brick tabloids in studying the permutation enumeration of the symmetric and other permutation groups, the methods we will describe in the rest of this monograph will not provide a better technique than [**51**] to use a power of q counting the number of cycles in a permutation. We will discuss this approach in an upcoming section.

The only work we have not discussed yet is the recent work of Langley and Remmel [**53**]. In this paper, Beck and Remmel's machinery is applied to the problem of the permutation enumeration of multiples of permutations. The same approach taken in [**53**] may be found in our Section 1.5. However, throughout this work we will give a large number of new generating functions for multiples of permutations. All of these generalizations will be refined by inversions, the major index, and a host of other statistics.

Also in [**53**], there are techniques to find generating functions for polynomials involving the length of the final increasing sequence in a permutation. The next section will provide a much better way to find information about the final increasing (and decreasing) sequence in a permutation by using our new basis $p_{n,\nu}$.

1.5. First extensions of Brenti's homomorphism

In this section we will extend the theorems in Section 1.3 in a few different ways. First, we will give a generalization to multiples of permutations. Then, Theorem 1.17 will be refined by inversion statistics, major index statistics, and statistics depending on the length of the final decreasing length of a permutation. The ideas in this section will be used throughout the rest of this monograph.

We begin our discussion on different ways to extend Brenti's homomorphism ξ with an example showing how to find permutation statistics for the m-fold product of the symmetric group S_n. Let the Cartesian product $S_n \times \cdots \times S_n$ (m times) be denoted by S_n^m. Define a homomorphism $\xi_2 : \Lambda \to \mathbb{Q}[x, y]$ such that

$$\xi_2^{f_1}(e_n) = \frac{(-1)^n}{(n!)^m} f_1(n) = \frac{(-1)^n(-y)(x-y)^{n-1}}{(n!)^m}.$$

We will give a relationship between $\xi_2^{f_1}$ and statistics over S_n^m. Notationally, $\xi_2^{f_1}$ depends on the integer m but this is not indicated. When more homomorphisms are defined, we will not explicitly denote the parameters on which they may depend; otherwise, we would be bogged down with the use of too many symbols in our equations.

For $(\sigma^1, \ldots, \sigma^m) \in S_n^m$, let $\mathrm{comdes}(\sigma^1, \ldots, \sigma^m)$ be the statistic counting the number of times σ^i has a descent occurring at the j^{th} place for every i. This is known as the number of common descents of $(\sigma^1, \ldots, \sigma^m)$. Let $\mathrm{oneris}(\sigma^1, \ldots, \sigma^m)$ be the number of times there exists an i such that σ^i has a rise occurring at the j^{th} place so that $\mathrm{comdes}(\sigma^1, \ldots, \sigma^m) + \mathrm{oneris}(\sigma^1, \ldots, \sigma^m) = n$ for all m-tuples of permutations of n.

THEOREM 1.23.

$$\sum_{n=0}^{\infty} \frac{t^n}{(n!)^m} \sum_{\sigma \in S_n^m} x^{\mathrm{comdes}(\sigma)} y^{\mathrm{oneris}(\sigma)} = \frac{x-y}{x - y\sum_{n=0}^{\infty} \frac{(x-y)^n t^n}{(n!)^m}}.$$

PROOF. By Theorem 1.12,

$$\begin{aligned} (n!)^m \xi_2^{f_1}(h_n) &= (n!)^m \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \prod_{i=1}^{\ell(\lambda)} \xi_2^{f_1}(e_{\lambda_i}) \\ &= \sum_{\lambda \vdash n} \binom{n}{\lambda}^m |B_{\lambda,(n)}| (-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell). \end{aligned} \tag{1.23}$$

Notice that by defining $\xi_2^{f_1}$ to have a factor of $(n!)^m$ in the denominator, m copies of the multinomial coefficient appear in (1.23).

To create objects to count (1.23), start with $T \in B_{\lambda,(n)}$. Use the m copies of the binomial coefficient in (1.23) to fill the bricks of T with m rows of decreasing sequences such that each row contains each of the integers $1, \ldots, n$ exactly once. The $(-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$ term labels the bricks in the same way as in Chapter 2 so that each brick has terminal cell labeled with y while each other cell is labeled with $-y$ or x. Below we give an example of one such object when $m = 2$.

$-y$	x	y	x	$-y$	$-y$	y	x	$-y$	x	x	y
10	9	1	8	7	3	2	12	11	6	5	4
12	8	2	10	5	3	1	11	9	7	6	4

Let the weight of such an object the product of all x and y labels. The weighted sum over all possible objects is equal to (1.23).

Now we apply a brick breaking/combining involution. Scan the bricks from left to right looking for the first occurrence of either a $-y$ or two consecutive bricks where each of the m rows of permutations contain a descent between the two bricks. If a $-y$ is scanned, break the brick into two at the violation of the $-y$ and change the $-y$ to y. If two consecutive bricks where each of the m rows of permutations contain a descent between the two bricks is scanned, combine the two bricks and change the resultant y in the middle of the new brick to $-y$. This is a weight-preserving sign-reversing involution. The fixed points under this involution may be interpreted as m elements of the symmetric group S_n in the obvious way; when this is done, the exponents on the powers of x and y register the appropriate statistics. Summing over all fixed points, we see that

$$(n!)^m \xi_2^{f_1}(h_n) = \sum_{\sigma \in S_n^m} x^{\mathrm{comdes}(\sigma^1,\ldots,\sigma^m)} y^{\mathrm{oneris}(\sigma^1,\ldots,\sigma^m)}.$$

Applying (1.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{(n!)^m} \sum_{\sigma \in S_n^m} x^{\mathrm{comdes}(\sigma)} y^{\mathrm{oneris}(\sigma)} &= \xi_2^{f_1}\left(\sum_{n=0}^{\infty} t^n h_n\right) \\ &= \left(\sum_{n=0}^{\infty} (-t)^n \xi_2^{f_1}(e_n)\right)^{-1} \\ &= \left(1 - y\sum_{n=1}^{\infty} \frac{(x-y)^{n-1} t^n}{(n!)^m}\right)^{-1},\end{aligned}$$

which may be simplified to the desired expression via multiplication by $(x-y)/(x-y)$. □

The only difference in the above proof and the proof of the theorems in Section 1.3 was the integer labeling of the bricks—instead of labeling the brick tabloid with one row of integers, brick tabloids were labeled with m rows of integers. This technique was first given in [**53**].

Other extensions of Section 1.3 involve refining Theorem 1.17 by other permutation statistics. To do this, standard notation from hypergeometric function theory will be used. For $n \geq 1$ and $\lambda \vdash n$, let

$$\begin{aligned}[n]_{p,q} &= \frac{p^n - q^n}{p-q} = p^{n-1}q^0 + \cdots + p^0 q^{n-1}, \\ [n]_{p,q}! &= [n]_{p,q} \cdots [1]_{p,q}, \qquad \text{and} \\ \begin{bmatrix} n \\ \lambda \end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[\lambda_1]_{p,q}! \cdots [\lambda_\ell]_{p,q}!}.\end{aligned}$$

be the p,q-analogues of n, $n!$, and $\binom{n}{\lambda}$, respectively. By convention, let $[0]_{p,q} = 0$ and $[0]_{p,q}! = 1$. In addition, let $(x,y;p,q)_0 = 1$ and

$$(x,y;p,q)_n = (x-y)(xp-yq)\cdots(xp^{n-1} - yq^{n-1}).$$

Suppose $r(t)$ is a function with power series $r(t) = \sum_{n=0}^{\infty} r_n t^n/n!$ for complex numbers r_n. A p,q-analogue for this function is defined by

$$r_{p,q}(t) = \sum_{n=0}^{\infty} r_n \frac{t^n}{[n]_{p,q}!} q^{\binom{n}{2}}.$$

For $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, define

$$\mathrm{coinv}(\sigma) = \sum_{i<j} \chi(\sigma_i < \sigma_j).$$

This is referred to as the number of co-inversions of σ. As in the case of inversions, this definition is valid for any sequence of integers. Notice that for $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, every pair of integers σ_i and σ_j must either register an inversion or co-inversion. Therefore, $\mathrm{inv}(\sigma) + \mathrm{coinv}(\sigma) = \binom{n}{2}$ for all $\sigma \in S_n$. The p,q-analogues defined above will help us keep track of inversions and co-inversions and the following lemma will aid the process.

LEMMA 1.24. *For positive integers $b_1, \ldots, b_\ell$ which sum to n,*

$$\begin{bmatrix} n \\ b_1, \ldots, b_\ell \end{bmatrix}_{p,q} = \sum_{r \in \mathcal{R}(1^{b_1}, \ldots, \ell^{b_\ell})} q^{\mathrm{inv}(r)} p^{\mathrm{coinv}(r)}$$

where $\mathcal{R}(1^{b_1}, \ldots, \ell^{b_\ell})$ is the set of rearrangements of b_1 1's, b_2 2's, etc.

PROOF. From the definition of $[n]_{p,q}$, $[n]_{p,q} = p^{n-1}[n]_{\frac{q}{p},1}$. We have

$$\begin{bmatrix} n \\ b_1, \ldots, b_\ell \end{bmatrix}_{q,p} = \frac{p^{(n-1)+\cdots+1}}{p^{(b_1-1)+\cdots+1} \cdots p^{(b_\ell-1)+\cdots+1}} \begin{bmatrix} n \\ b_1, \ldots, b_\ell \end{bmatrix}_{\frac{q}{p},1}.$$

Carlitz showed in [**19**] that $\begin{bmatrix} n \\ \lambda \end{bmatrix}_{q,1} = \sum_{r \in \mathcal{R}(1^{b_1}, \ldots, \ell^{b_\ell})} q^{\mathrm{inv}(r)}$. Therefore, the above equation may be written to read

$$p^{\binom{n}{2}-\binom{b_1}{2}-\cdots-\binom{b_\ell}{2}} \sum_{r \in \mathcal{R}(1^{b_1}, \ldots, \ell^{b_\ell})} \frac{q^{\mathrm{inv}(r)}}{p^{\mathrm{inv}(r)}} = \sum_{r \in \mathcal{R}(1^{b_1}, \ldots, \ell^{b_\ell})} q^{\mathrm{inv}(r)} p^{\binom{n}{2}-\binom{b_1}{2}-\cdots-\binom{b_\ell}{2}-\mathrm{inv}(r)}.$$

The lemma follows by noting that

$$\mathrm{coinv}(r) = \binom{n}{2} - \binom{b_1}{2} - \cdots - \binom{b_\ell}{2} - \mathrm{inv}(r). \qquad \square$$

At this point we are prepared to commence our discussion of refining Theorem 1.17 by the inversion and co-inversion statistics. Define $\xi_3^{f_1}$ on the n^{th} elementary symmetric function such that

$$\xi_3^{f_1}(e_n) = \frac{(-1)^n}{[n]_{p,q}!} q^{\binom{n}{2}} f_1(n) = \frac{(-1)^n}{[n]_{p,q}!} q^{\binom{n}{2}} (-y)(x-y)^{n-1}.$$

This homomorphism will be used in the proof of Theorem 1.25 below.

THEOREM 1.25.

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_{q,p}!} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} = \frac{x-y}{x - y\mathbf{e}_{p,q}^{t(x-y)}}.$$

PROOF. We begin as usual by exploiting the relationship between the homogeneous symmetric functions and the elementary symmetric functions. We have

$$\begin{aligned}
[n]_{p,q}! \xi_3^{f_1}(h_n) &= [n]_{p,q}! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \prod_{i=1}^{\ell(\lambda)} \xi_3^{f_1}(e_{\lambda_i}) \\
&= \sum_{\lambda \vdash n} \begin{bmatrix} n \\ \lambda \end{bmatrix}_{p,q} q^{\binom{\lambda_1}{2}+\cdots+\binom{\lambda_\ell}{2}} |B_{\lambda,(n)}| (-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell). \qquad (1.24)
\end{aligned}$$

The multinomial coefficient and the power of q in the above equation arise from the p,q-analogues in the definition of $\xi_3^{f_1}$. Equation (1.24) will be interpreted as a sum of signed weighted objects on which to perform involutions.

Let us start with a brick tabloid of shape (n) and type λ for some $\lambda \vdash n$ and weight each brick such that y appears in the last cell and x or $-y$ in every other cell. This uses the summand, $|B_{\lambda,(n)}|$, and $(-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$ terms in (1.24).

The powers of p and q are the only components in (1.24) which we need to use in forming combinatorial objects in which to perform an involution.

For a brick tabloid of shape (n) and type λ, let $b_1,\ldots,b_\ell$ be the lengths of the bricks when read from left to right in the tabloid T. The numbers $b_1,\ldots,b_\ell$ are a rearrangement of the parts of λ. For every $r \in \mathcal{R}(1^{b_1},\ldots,\ell^{b_\ell})$, let τ_r be the permutation in S_n formed by numbering from right to left all the 1's in r with 1 to b_1, all the 2's in r with b_1+1 to b_2, etc. Then fill the cells of T with the integers found in the inverse permutation τ_r^{-1}.

For example, suppose T is a brick tabloid of shape (n) and type $\lambda = (2,3,5)$ where reading from left to right the bricks come in the order $3,5,2$. Let $r = 2\ 2\ 3\ 1\ 2\ 3\ 1\ 2\ 2\ 1\ 3\ 1$ be an element in $\mathcal{R}(1^4,2^5,3^3)$. The construction of τ_r and τ_r^{-1} is displayed below.

		1	2	3	4	5	6	7	8	9	10	11	12
r	$=$	2	2	3	1	2	3	1	2	2	1	3	1
τ_r	$=$	9	8	12	4	7	11	3	6	5	2	10	1
τ_r^{-1}	$=$	12	10	7	4	9	8	5	2	1	11	6	3

By construction, τ_r^{-1} has decreasing sequences of lengths of $b_1,\ldots,b_\ell$ because the first b_1 integers in τ_r^{-1} register the positions of the 1's, the next b_2 integers register the positions of the 2's, etc. From the construction of r, we have

$$\mathrm{inv}(\tau_r^{-1}) = \mathrm{inv}(\tau_r) = \mathrm{inv}(r) + \binom{b_1}{2} + \cdots + \binom{b_\ell}{2} = \mathrm{inv}(r) + \binom{\lambda_1}{2} + \cdots + \binom{\lambda_\ell}{2}$$

and

$$\mathrm{coinv}(\tau_r^{-1}) = \mathrm{coinv}(\tau_r) = \mathrm{coinv}(r).$$

Therefore, Lemma 1.24 allows for the weighting of a brick tabloid with both $q^{\mathrm{inv}(\tau_r^{-1})}$ and $p^{\mathrm{coinv}(\tau_r^{-1})}$ to account for everything in (1.24). Specifically, let us record these powers of q and p in the brick tabloids by placing in each cell a power of q registering the number of smaller integers to the right and a power of p registering the number of larger integers to the right. For example, below we show one object counted by (1.24).

x	x	$-y$	y	x	$-y$	$-y$	x	y	$-y$	$-y$	y
$q^{11}p^0$	q^9p^1	q^6p^3	q^3p^5	q^6p^1	q^5p^1	q^3p^2	q^1p^3	q^0p^3	q^2p^0	q^1p^0	q^0p^0
12	10	7	4	9	8	5	2	1	11	6	3

Define the weight of an object as the product of the powers of x, y, p and q appearing in the cells. It follows that the sum of weights over all possible combinatorial objects is equal to (1.24).

Perform the brick breaking and combining involution I_ξ in Section 1.3 on this collection of objects, ignoring the powers of p and q. This involution is weight-preserving and sign-reversing. The fixed points correspond to elements of S_n where the x-weight counts descents, y-weight counts rises, p-weight counts co-inversions, and q-weight counts inversions.

Therefore, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_{q,p}!} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} = \xi_3^{f_1}\left(\sum_{n=0}^{\infty} t^n h_n\right) = \left(\sum_{n=0}^{\infty} (-t)^n \xi_3^{f_1}(e_n)\right)^{-1},$$

from which the statement of the theorem follows from routine symbolic manipulation. □

The ideas in Theorem 1.23 and the ideas in Theorem 1.25 may be combined. That is, we may define a homomorphism mapping e_n to

$$\frac{(-1)^n}{[n]_{p_1,q_1}! \cdots [n]_{p_m,q_m}!} q_1^{\binom{n}{2}} \cdots q_m^{\binom{n}{2}} f_1(n).$$

This definition allows for powers of p_i and q_i to register inversions and co-inversions on each of the m rows of permutations. Then, the generating function

$$\begin{aligned}(1.25)\quad & \sum_{n=0}^{\infty} \frac{t^n}{[n]_{p,q}!} \\ & \times \sum_{\sigma=(\sigma^1,\cdots,\sigma^m) \in S_n^m} x^{\mathrm{comdes}(\sigma)} y^{\mathrm{oneris}(\sigma)} q_1^{\mathrm{inv}(\sigma^1)} p_1^{\mathrm{coinv}(\sigma^1)} \cdots q_m^{\mathrm{inv}(\sigma^m)} p_m^{\mathrm{coinv}(\sigma^m)} \\ & = \left(\sum_{n=0}^{\infty} \frac{t^n}{[n]_{p_1,q_1}! \cdots [n]_{p_m,q_m}!} q_1^{\binom{n}{2}} \cdots q_m^{\binom{n}{2}} (-y)(x-y)^{n-1}\right)^{-1}\end{aligned}$$

follows from (1.4).

More generating functions for multivariate permutation statistics can be found with the aid of different homomorphisms. Let $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ and define

$$\begin{aligned}\mathrm{comaj}(\sigma) &= \sum_{i=1}^{n} i\, \chi(\sigma_{i+1} > \sigma_i), \\ \mathrm{rlmaj}(\sigma) &= \sum_{i=1}^{n} (n-i)\, \chi(\sigma_{i+1} < \sigma_i), \qquad \text{and} \\ \mathrm{rlcomaj}(\sigma) &= \sum_{i=1}^{n} (n-i)\, \chi(\sigma_{i+1} > \sigma_i),\end{aligned}$$

where, by convention, $\sigma_{n+1} = n+1$. These are known as the co-major index, right-to-left major index, and co-right-to-left major index statistics for the symmetric group. For any $\sigma \in S_n$, $\mathrm{maj}(\sigma) + \mathrm{comaj}(\sigma) = \binom{n+1}{2}$ and $\mathrm{rlmaj}(\sigma) + \mathrm{rlcomaj}(\sigma) = \binom{n+1}{2}$.

Let j be a positive integer and define $\xi_4^{f_1}$ as a homomorphism on the ring of symmetric functions such that

$$\xi_4^{f_1}(e_n) = (-1)^n q^{-nj} \begin{bmatrix} j+n \\ n \end{bmatrix}_{p,q} f_1(n) = (-1)^n q^{-nj} \begin{bmatrix} j+n \\ n \end{bmatrix}_{p,q} (-y)(x-y)^{n-1}.$$

THEOREM 1.26.

$$\sum_{n=0}^{\infty} \frac{t^n}{(u,v;q,p)_{n+1}} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma^{-1})} y^{\mathrm{ris}(\sigma^{-1})} u^{\mathrm{des}(\sigma)} v^{\mathrm{ris}(\sigma)} q^{\mathrm{maj}(\sigma)} p^{\mathrm{comaj}(\sigma)}$$
$$= \sum_{j=1}^{\infty} \frac{v^j(x-y)}{u^{j+1}\left(x - y\sum_{n=0}^{\infty}(tp(x-y)/q^j)^n \begin{bmatrix} j+n-1 \\ n \end{bmatrix}_{p,q}\right)}.$$

PROOF. We have

$$\xi_4^{f_1}(h_n) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \prod_{i=1}^{\ell(\lambda)} \xi_4^{f_1}(e_{\lambda_i})$$
$$= \sum_{\lambda \vdash n} \left(\prod_{i=1}^{\ell(\lambda)} q^{-\lambda_i j} \begin{bmatrix} j+\lambda_i \\ \lambda_i \end{bmatrix}_{p,q} \right) |B_{\lambda,(n)}| (-1)^{\ell(\lambda)} f_1(\lambda_1)\cdots f_1(\lambda_\ell). \tag{1.26}$$

Start to count (1.26) with $T \in B_{\lambda,(n)}$ labeled with powers of x and $-y$ in every cell not ending a brick and y in every terminal cell.

Notice that for any $r \in \mathcal{R}(0^j, 1^{\lambda_i})$, $\mathrm{inv}(r) + \mathrm{coinv}(r) = \lambda_i j$. By Lemma 1.24, we have

$$q^{-\lambda_i j}\begin{bmatrix} j+\lambda_i \\ \lambda_i \end{bmatrix}_{p,q} = q^{-\lambda_i j} \sum_{r \in \mathcal{R}(0^j,1^{\lambda_i})} p^{\mathrm{inv}(r)} q^{\mathrm{coinv}(r)}$$
$$= q^{-\lambda_i j} \sum_{r \in \mathcal{R}(0^j,1^{\lambda_i})} p^{\mathrm{inv}(r)} q^{\lambda_i j - \mathrm{inv}(r)}$$
$$= \sum_{r \in \mathcal{R}(0^j,1^{\lambda_i})} (p/q)^{\mathrm{inv}(r)}. \tag{1.27}$$

For $r \in \mathcal{R}(0^j, 1^{\lambda_i})$, let b_k be the number of 0's after the $(\lambda_i - k)^{\mathrm{th}}$ 1. Then, (1.27) is equal to

$$\sum_{0 \le b_1 \le \cdots \le b_{\lambda_i} \le j} (p/q)^{b_1 + \cdots + b_{\lambda_i}}.$$

Therefore, with the product of p,q-binomial coefficients in (1.26), place a weakly increasing sequence of positive integers with maximum value j in each brick and mark each cell with p/q to the power of the integer in that cell. For example, below we give an example of one such object when $j = 3$.

x	$-y$	$-y$	y	$-y$	y	$-y$	x	x	y	x	y
p^0q^0	p^1q^{-1}	p^2q^{-2}	p^3q^{-3}	p^0q^0	p^1q^{-1}	p^0q^0	p^0q^0	p^2q^{-2}	p^3q^{-3}	p^1q^{-1}	p^3q^{-3}
0	1	2	3	0	1	0	0	2	3	1	3

Define the weight as the product of the x, y, q, and p labels. It follows that the weighted sum over all objects formed in this manner is equal to the sum in (1.26).

Perform the brick breaking/combining involution where bricks are scanned from left to right looking for a $-y$ or two consecutive bricks where there is a weak increase. Bricks are either broken or combined according to the situation. The remaining fixed points have positive sign, strict decreases between bricks, a power of x for every weak increase, and a power of y for every strict decrease.

Suppose r is the word with letters $0,\ldots,j$ taken from a fixed point. Let τ_r^{-1} be the permutation formed by labeling the j's, then the $j-1$'s, and so on, from right to left with integers from 1 to n. Then the first integers in τ_r register the positions of the j's, the next integers register the positions of the $j-1$'s, and in general, at the i^{th} iteration, the integers register the positions of the $j-i$'s. Let c be the rearrangement of $0,\ldots,j$ keeping track of how many 0's, …, j's were in r by sorting r in decreasing order. Finally, let a be the sequence with part i equal to k if and only if there is an decrease of size k in the sequence c after place i. † For example, suppose that $r = 0\ 1\ 2\ 3\ 0\ 1\ 0\ 0\ 2\ 3\ 1\ 3$. The table below shows τ_r^{-1}, τ_r,c, and a:

		1	2	3	4	5	6	7	8	9	10	11	12
r	=	0	1	2	3	0	1	0	0	2	3	1	3
τ_r^{-1}	=	12	8	5	3	11	7	10	9	4	2	6	1
τ_r	=	12	10	4	9	3	11	6	2	8	7	5	1
c	=	3	3	3	2	2	1	1	1	0	0	0	0
a	=	0	0	1	0	1	0	0	1	0	0	0	0

From this construction, the x weight in a fixed point is $x^{\operatorname{des}(\tau_r^{-1})}$ and the y weight in a fixed point is $y^{\operatorname{ris}(\tau_r^{-1})}$. Furthermore, $c_i = a_i + \cdots + a_n$, so the p/q weight of the fixed point corresponding to r is given by $c_1 + \cdots + c_n = 1a_1 + 2a_2 + \cdots + na_n$. The maximum value in r is $c_1 = a_1 + \cdots + a_n$, where $a_i \geq \chi\left(\tau_r(i) < \tau_r(i+1)\right)$ for $i = 1,\ldots,n-1$, and $a_n \geq 0$. Therefore, the sum over all possible fixed points under the brick breaking involution—and hence $\xi_4^{f_1}(h_n)$—is equal to

$$\begin{aligned}
&\sum_{\substack{a_i \geq \chi(\tau_r(i) < \tau_r(i+1)) \\ a_1 + \cdots + a_n \leq j}} x^{\operatorname{des}(\tau_r^{-1})} y^{\operatorname{ris}(\tau_r^{-1})} (p/q)^{1a_1 + \cdots + na_n} \\
&= \sum_{a_i \geq \chi(\tau_r(i) < \tau_r(i+1))} x^{\operatorname{des}(\tau_r^{-1})} y^{\operatorname{ris}(\tau_r^{-1})} (p/q)^{1a_1 + \cdots + na_n} (v/u)^{a_1 + \cdots + a_n} \Bigg|_{(v/u)^{\leq j}} \\
&= \sum_{\sigma \in S_n} x^{\operatorname{des}(\sigma^{-1})} y^{\operatorname{ris}(\sigma^{-1})} \sum_{a_i \geq \chi(\sigma_i < \sigma_{i+1})} \prod_{i=1}^{n} (p/q)^{ia_i} (v/u)^{a_i} \Bigg|_{(v/u)^{\leq j}}
\end{aligned}$$

†The dependence of c and a on r is not indicated to unclutter notation.

where $expression|_{u\leq j}$ means to sum the coefficients of u^i in *expression* for $i = 0,\ldots,j$. Our string of equalities is equal to

$$\begin{aligned}
&\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma^{-1})}y^{\mathrm{ris}(\sigma^{-1})}\prod_{i=1}^{n}\sum_{a_i\geq\chi(\sigma_i<\sigma_{i+1})}\left((v/u)(p/q)^i\right)^{a_i}\Bigg|_{(v/u)^{\leq j}}\\
&\quad=\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma^{-1})}y^{\mathrm{ris}(\sigma^{-1})}\frac{(v/u)^{\sum_{i=1}^n\chi(\sigma_i<\sigma_{i+1})}(p/q)^{\sum_{i=1}^n i\chi(\sigma_i<\sigma_{i+1})}}{(1-(v/u)(p/q))\cdots(1-(v/u)(p/q)^n)}\Bigg|_{(v/u)^{\leq j}}\\
&\quad=\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma^{-1})}y^{\mathrm{ris}(\sigma^{-1})}\frac{(v/u)^{\mathrm{ris}(\sigma)-1}(p/q)^{\mathrm{comaj}(\sigma)-n}u^{n+1}q^{\binom{n+1}{2}}}{(u-v)(uq-vp)\cdots(uq^n-vp^n)}\Bigg|_{(v/u)^{j}}\\
&\quad=\frac{u^2q^n}{vp^n}\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma^{-1})}y^{\mathrm{ris}(\sigma^{-1})}\frac{u^{\mathrm{des}(\sigma)}v^{\mathrm{ris}(\sigma)}q^{\mathrm{maj}(\sigma)}p^{\mathrm{comaj}(\sigma)}}{(u,v;q,p)_{n+1}}\Bigg|_{(v/u)^{j}}.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n}{(u,v;q,p)_{n+1}}\frac{u^2q^n}{vp^n}\sum_{\sigma\in S_n}x^{\mathrm{des}(\sigma^{-1})}y^{\mathrm{ris}(\sigma^{-1})}u^{\mathrm{des}(\sigma)}v^{\mathrm{ris}(\sigma)}q^{\mathrm{maj}(\sigma)}p^{\mathrm{comaj}(\sigma)}\\
&\qquad=\sum_{j=0}^{\infty}(v/u)^j\xi_4^{f_1}\left(\sum_{n=0}^{\infty}h_nt^n\right)\\
&\qquad=\sum_{j=0}^{\infty}(v/u)^j\left(1+\sum_{n=1}^{\infty}t^n(-y)(x-y)^{n-1}q^{-nj}\begin{bmatrix}j+n\\ n\end{bmatrix}_{p,q}\right)^{-1},
\end{aligned}$$

where the last line follows from the definition of $\xi_4^{f_1}$ on the elementary symmetric function e_n. Taking t as $t(p/q)$ in the extremes of the above string of inequalities and simplifying the result provides the statement of the theorem. $\square$

The same ideas in the proof of Theorem 1.26 can just as easily prove that

$$\begin{aligned}
(1.28)\quad&\sum_{n=0}^{\infty}\frac{t^n}{(u,v;q,p)_{n+1}}\sum_{\sigma\in S_n}x^{\mathrm{des}(\sigma^{-1})}y^{\mathrm{ris}(\sigma^{-1})}u^{\mathrm{des}(\sigma)}v^{\mathrm{ris}(\sigma)}q^{\mathrm{rlmaj}(\sigma)}p^{\mathrm{rlcomaj}(\sigma)}\\
&\qquad=\sum_{j=1}^{\infty}\frac{v^jq(x-y)}{u^{j+1}p\left(x-y\sum_{n=0}^{\infty}(t(x-y)/q^{j-1})^n\begin{bmatrix}j+n-1\\ n\end{bmatrix}_{p,q}\right)}.
\end{aligned}$$

Only minor adjustments from the proof of Theorem 1.26 are needed to show (1.28) above is true. Instead of labeling the bricks in a brick tabloid with increasing sequence of positive numbers, write the positive numbers in decreasing order. Apply a similar involution as in the proof of Theorem 1.26. Then, when forming the permutation τ_r^{-1} from the word r in a fixed point, first label the 0's from right to left, then the 1's from right to left, etc., with the integers $1,\ldots,n$. Let c be the rearrangement r coming from a fixed point written in increasing order and a the sequence indicating rises in c. Below we gives an example of this alternative labeling when $r = 3\ 2\ 1\ 0\ 1\ 0\ 3\ 2\ 0\ 0\ 3\ 1$ is the sequence of positive integers read off a fixed point.

		1	2	3	4	5	6	7	8	9	10	11	12
r	$=$	3	2	1	0	1	0	3	2	0	0	3	1
τ_r^{-1}	$=$	12	9	7	4	6	3	11	8	2	1	10	5
τ_r	$=$	10	9	6	4	12	5	3	8	2	11	7	1
c	$=$	0	0	0	0	1	1	1	2	2	3	3	3
a	$=$	0	0	0	0	1	0	0	1	0	1	0	0

With these adjustments, the rest of the proof of Theorem 1.26 follows through in a similar manner. This relabeling enables a generating function for right/left versions of the major index statistics to be stated as in (1.28).

There is another path to finding the generating function for the left hand side of the equation in Theorem 1.26. It uses a slightly different homomorphism as $\xi_4^{f_1}$. Define $\xi_5^{f_1}$ as a homomorphism on the elementary symmetric functions by

$$\begin{aligned}\xi_5^{f_1}(e_n) &= (-1)^n p^{\binom{n}{2}-nj} q^{\binom{n}{2}} \begin{bmatrix} j+1 \\ n \end{bmatrix}_{p,q} f_1(n) \\ &= (-1)^n p^{\binom{n}{2}-nj} q^{\binom{n}{2}} \begin{bmatrix} j+1 \\ n \end{bmatrix}_{p,q} (-y)(x-y)^{n-1}.\end{aligned}$$

THEOREM 1.27.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{(u,v;q,p)_{n+1}} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma^{-1})} y^{\mathrm{ris}(\sigma^{-1})} u^{\mathrm{des}(\sigma)} v^{\mathrm{ris}(\sigma)} q^{\mathrm{maj}(\sigma)} p^{\mathrm{comaj}(\sigma)} \\ = \sum_{j=0}^{\infty} \frac{u^j}{v^{j+1}} \left(\frac{y-x}{x - y\sum_{n=0}^{\infty} (t(y-x)/p^j)^n p^{\binom{n}{2}} q^{\binom{n}{2}} \left[\begin{smallmatrix} j+1 \\ n \end{smallmatrix}\right]_{p,q}} \right).\end{aligned}$$

PROOF. Applying ξ_5 on h_n, we have from Theorem 1.12 that

(1.29)
$$\xi_5^{f_1}(h_n) = \sum_{\lambda \vdash n} \left(\prod_{i=1}^{\ell(\lambda)} p^{\binom{\lambda_i}{2}-\lambda_i j} q^{\binom{\lambda_i}{2}} \begin{bmatrix} j+1 \\ \lambda_i \end{bmatrix}_{p,q} \right) |B_{\lambda,(n)}| (-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell).$$

To count (1.29), start with $T \in B_{\lambda,(n)}$ labeled with powers of x and $-y$ in every cell not ending a brick and y in every terminal cell.

The powers of p and q in (1.29) give us a slightly different filling of the brick tabloids as that in Theorem 1.26. We have

$$p^{\binom{\lambda_i}{2}-\lambda_i j} q^{\binom{\lambda_i}{2}} \begin{bmatrix} j+1 \\ \lambda_i \end{bmatrix}_{p,q} = p^{-\lambda_i(j+1-\lambda_i)} (q/p)^{\binom{\lambda_i}{2}} \begin{bmatrix} j+1 \\ \lambda_i \end{bmatrix}_{p,q},$$

which, via the same steps found in Theorem 1.26 applied to (1.27), is equal to

$$(q/p)^{\binom{\lambda_i}{2}} \sum_{0 \le b_1 \le \cdots \le b_{\lambda_i} \le j+1-\lambda_i} (q/p)^{b_1+\cdots+b_{\lambda_i}} = \sum_{0 \le b_1 < \cdots < b_{\lambda_i} \le j+1-\lambda_i} (q/p)^{b_1+\cdots+b_{\lambda_i}}.$$

Therefore, with the product of p,q-binomial coefficients in (1.29), place a strictly increasing sequence of positive integers with maximum value j in each brick and mark each cell with q/p to the power of the integer in that cell. Define the weight as the product of the x, y, q, and p labels. It follows that the weighted sum over all objects formed in this manner is equal to the sum in (1.29).

Perform the brick breaking/combining involution where bricks are scanned from left to right looking for a $-y$ or two consecutive bricks where there is a strict increase. Bricks are either broken or combined according to the situation. The remaining fixed points have positive sign, weak decreases between bricks, a power of x for every strict increase, and a power of y for every weak decrease. An example of such a fixed point when $j = 3$ is found below.

x	x	y	y	x	y	y	x	y	x	x	y
p^0q^0	p^1q^{-1}	p^3q^{-3}	p^3q^{-3}	p^0q^0	p^1q^{-1}	p^0q^0	p^0q^0	p^2q^{-2}	p^1q^{-1}	p^2q^{-2}	p^3q^{-3}
0	1	3	3	0	1	0	0	2	1	2	3

Suppose r is the word with letters $0, \ldots, j$ taken from a fixed point. Let τ_r^{-1} be the permutation formed by labeling the j's, then the $j-1$'s, and so on, from left to right with integers from 1 to n. Then the first integers in τ_r register the positions of the j's, the next integers register the positions of the $j-1$'s, and in general, in the i^{th} iteration, the integers in τ_r register the positions of the $j-i$'s. Let c be the rearrangement of $0, \ldots, j$ keeping track of how many 0's, $\ldots$, j's were in r by sorting r in decreasing order. Finally, let a be the sequence with part i equal to k if and only if there is an decrease of size k in the sequence c after place i. For example, suppose that $r = 0\ 1\ 3\ 3\ 0\ 1\ 0\ 0\ 2\ 1\ 2\ 3$. Below we show τ_r^{-1}, τ_r, c, and a:

		1	2	3	4	5	6	7	8	9	10	11	12
r	=	0	1	3	3	0	1	0	0	2	1	2	3
τ_r^{-1}	=	9	6	1	2	10	7	11	12	4	8	5	3
τ_r	=	3	4	12	9	11	2	6	10	1	5	7	8
c	=	3	3	3	2	2	1	1	1	0	0	0	0
a	=	0	0	1	0	1	0	0	1	0	0	0	0

From this construction, the x weight in a fixed point is $x^{\text{des}(\tau_r^{-1})}$ and the y weight in a fixed point is $y^{\text{ris}(\tau_r^{-1})}$. Furthermore, $c_i = a_i + \cdots + a_n$, so the q/p weight of the fixed point corresponding to r is given by $c_1 + \cdots + c_n = 1a_1 + 2a_2 + \cdots + na_n$. The maximum value in r is $c_1 = a_1 + \cdots + a_n$, where $a_i \geq \chi(\tau_r(i) > \tau_r(i+1))$ for $i = 1, \ldots, n-1$, and $a_n \geq 0$. Therefore, the sum over all possible fixed points under the brick breaking involution—and hence $\xi_5^{f_1}(h_n)$—is equal to

$$\sum_{\substack{a_i \geq \chi(\tau_r(i) > \tau_r(i+1)) \\ a_1 + \cdots + a_n \leq j}} x^{\text{des}(\tau_r^{-1})} y^{\text{ris}(\tau_r^{-1})} (q/p)^{1a_1 + \cdots + na_n}$$

$$= \sum_{a_i \geq \chi(\tau_r(i) > \tau_r(i+1))} x^{\text{des}(\tau_r^{-1})} y^{\text{ris}(\tau_r^{-1})} (q/p)^{1a_1 + \cdots + na_n} (u/v)^{a_1 + \cdots + a_n} \Bigg|_{(u/v)^{\leq j}}.$$

In the same way as in the proof of Theorem 1.26, this expression is equal to

$$\sum_{\sigma \in S_n} x^{\text{des}(\sigma^{-1})} y^{\text{ris}(\sigma^{-1})} \frac{(u/v)^{\text{des}(\sigma)} (q/p)^{\text{maj}(\sigma)} v^{n+1} p^{\binom{n+1}{2}}}{(v-u)(vp-uq)\cdots(vp^n - uq^n)} \Bigg|_{(u/v)^j}$$

$$= v \sum_{\sigma \in S_n} x^{\text{des}(\sigma^{-1})} y^{\text{ris}(\sigma^{-1})} \frac{u^{\text{des}(\sigma)} v^{\text{ris}(\sigma)} q^{\text{maj}(\sigma)} p^{\text{comaj}(\sigma)}}{(v, u; p, q)_{n+1}} \Bigg|_{(u/v)^j}.$$

Using the fact that $(v,u;p,q)_{n+1} = (-1)^{n+1}(u,v;q,p)_{n+1}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{v(-1)^{n+1}t^n}{(u,v;q,p)_{n+1}} &\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma^{-1})}y^{\mathrm{ris}(\sigma^{-1})}u^{\mathrm{des}(\sigma)}v^{\mathrm{ris}(\sigma)}q^{\mathrm{maj}(\sigma)}p^{\mathrm{comaj}(\sigma)}\\ &= \sum_{j=0}^{\infty}(u/v)^j \xi_5^{f_1}\left(\sum_{n=0}^{\infty} h_n t^n\right)\\ &= \sum_{j=0}^{\infty}(u/v)^j \left(1+\sum_{n=1}^{\infty} t^n(-y)(x-y)^{n-1}p^{\binom{n}{2}-nj}q^{\binom{n}{2}}\begin{bmatrix} j+1\\ n\end{bmatrix}_{p,q}\right)^{-1},\end{aligned}$$

where the last line follows from the definition of $\xi_5^{f_1}$ on the elementary symmetric function e_n. This expression simplifies to the statement of the theorem. □

Filling the bricks in Theorem 1.27 in decreasing order can give an alternate expression for the generating function in (1.28). Furthermore, notice that the right hand sides of the expressions in Theorem 1.26 and Theorem 1.27 are equal to each other!

Again, these homomorphisms may be combined to produce results for multiples of permutations. To provide one specific example, suppose we wanted a generating function for

$$\begin{aligned}(1.30)\qquad \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma,\tau\in S_n} & x^{\mathrm{comdes}(\sigma^{-1},\tau^{-1})}y^{\mathrm{oneris}(\sigma^{-1},\tau^{-1})}\\ &\times u_1^{\mathrm{des}(\sigma)}v_1^{\mathrm{ris}(\sigma)}q_1^{\mathrm{maj}(\sigma)}p_1^{\mathrm{comaj}(\sigma)}u_2^{\mathrm{des}(\tau)}v_2^{\mathrm{ris}(\tau)}q_2^{\mathrm{maj}(\tau)}p_2^{\mathrm{comaj}(\tau)}.\end{aligned}$$

To find such an object, we may define a homomorphism mapping e_n to

$$(-1)^n q_1^{-nj}\begin{bmatrix} j+n\\ n\end{bmatrix}_{p_1,q_1} p_2^{\binom{n}{2}}q_2^{\binom{n}{2}}\begin{bmatrix} \ell+1\\ n\end{bmatrix}_{p_2,q_2}(-y)(x-y)^{n-1}.$$

When this homomorphism is applied to the homogeneous symmetric functions, the powers of q_1 and p_1 may be used as in the proof of Theorem 1.26 to decorate one row of weakly increasing numbers in a brick tabloid while the powers of q_2 and p_2 may be used as in the proof of Theorem 1.27 to decorate a second row of strictly increasing numbers. Applying the same sort of involution as found in the proof of Theorem 1.23, fixed points correspond to permutations in S_n with indeterminates registering the correct statistics found in (1.30). A generating function follows from an appropriate application of (1.4).

Of course there are many other examples like (1.30) which we could have chosen to display. Our selection was designed to appear similar to a result appearing in the literature—taking $y = v_1 = q_1 = u_2 = q_2 = 1$ in (1.30) gives the main result in a paper by Fedou and Rawlings [**34**].

At this point, we have refined Theorem 1.17 by both inversion statistics and major index statistics. To conclude this section, we will refine Theorem 1.17 by two more statistics examining the length of the final increasing or decreasing segment in a permutation of n. To do this, we will utilize our new basis $p_{n,\nu}$ for some function ν for the first time in this monograph.

Let $\mathrm{fd}(\sigma)$ be the length of the final decreasing sequence in $\sigma \in S_n$. For a positive integer j, let ν_1 be the function mapping on the nonnegative integers such that for $n \geq 1$,

$$\nu_1(n) = \begin{cases} 0 & \text{if } n < j, \\ \left(\dfrac{x}{x-y}\right)^{j-1} & \text{if } n = j, \text{ and} \\ \dfrac{-y}{x-y}\left(\dfrac{x}{x-y}\right)^{j-1} & \text{if } n > j. \end{cases}$$

This weight to assign the last bricks in a brick tabloid of shape λ will be used in conjunction with the homomorphism ξ^{f_1} to prove Theorem 1.28 below.

THEOREM 1.28.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} z^{\mathrm{fd}(\sigma)} = \left(\frac{x-y}{x - y\mathbf{e}^{t(x-y)}}\right)\left(1 + (1-z)\frac{y(\mathbf{e}^{xzt} - \mathbf{e}^{t(x-y)})}{x-y-xz}\right).$$

PROOF. By Theorem 1.15,

$$n!\xi^{f_1}(p_{n,\nu_1}) = n! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_{\nu_1}(B_{\lambda,(n)}) \prod_{i=1}^{\ell(\lambda)} \xi^{f_1}(e_{\lambda_i})$$

$$= \sum_{\lambda \vdash n} \binom{n}{\lambda} w_{\nu_1}(B_{\lambda,(n)})(-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell). \tag{1.31}$$

The weight from ν_1 assures us that the last brick in $T \in B_{\lambda,(n)}$ must be at least j cells in length; otherwise the weight on such a T would be 0. So, let us start to count (1.31) with $T \in B_{\lambda,(n)}$ where the last brick must be at least j cells in length. Use the binomial coefficient in (1.31) to fill the cells of T with a permutation of n such that each brick contains a decreasing sequence. Use the $(-1)^{\ell(\lambda)} f_1(\lambda_1) \cdots f_1(\lambda_\ell)$ term to weight the bricks such that either an x or $-y$ appears in every nonterminal cell and y appears in the terminal cell of every brick.

Suppose the last brick in T is of length j. The weight on the last brick given by ν_1 is $x^{j-1}/(x-y)^{j-1}$. This enables us to replace our choice of either x or $-y$ in the nonterminal cells of T with x. If the last brick in T is longer than j cells, the weight on the last brick given by ν_1 is $x^{j-1}(-y)/(x-y)^j$. This enables us to replace the last j choices for x or $-y$ with one $-y$ followed by $j-1$ x's. One object which may be formed in this manner is found below.

y	x	$-y$	y	$-y$	x	$-y$	x	x	x	x	y
5	8	7	3	12	11	10	9	6	4	2	1

The weighted sum over all such objects is equal to the sum in (1.31).

Apply the brick breaking/combining involution featured in Section 1.3 where brick are scanned from left to right for the first occurrence of either a $-y$ or two consecutive bricks with a decrease in the integer labeling between them. If the last brick in T is longer than j cells, a $-y$ must appear in the $(j+1)^{\text{th}}$ cell counting from right to left. Thus, a fixed point under this involution must end in a brick with exactly j cells. This shows fixed points correspond to those $\sigma \in S_n$ with $\mathrm{fd}(\sigma) = j$.

Therefore, we have that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)}z^{\mathrm{fd}(\sigma)} &= 1+\sum_{j=1}^{\infty}z^j\left(\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)}z^{\mathrm{fd}(\sigma)}\Bigg|_{z^j}\right)\\ &= 1+\sum_{j=1}^{\infty}z^j\xi^{f_1}\left(\sum_{n=1}^{\infty}t^n p_{n,\nu_1}\right)\\ &= 1+\sum_{j=1}^{\infty}z^j\xi^{f_1}\left(\frac{\sum_{n=1}^{\infty}(-1)^{n-1}\nu_1(n)e_nt^n}{1+\sum_{n=1}^{\infty}e_n(-t)^n}\right),\end{aligned}$$

which, by the definition of ν_1 and ξ^{f_1}, is equal to

$$1+\sum_{j=1}^{\infty}z^j\frac{yx^{j-1}\frac{t^j}{j!}-y^2\sum_{n=j+1}^{\infty}\frac{x^{j-1}}{(x-y)^j}(-y)(x-y)^{n-1}\frac{t^n}{n!}}{1-y\sum_{n=1}^{\infty}(x-y)^{n-1}\frac{t^n}{n!}}.$$

Through routine manipulations, the above expression may be simplified to look like the statement of the theorem. □

Let $\mathrm{fi}(\sigma)$ be the length of the final increasing segment of $\sigma\in S_n$. A clever choice of homomorphism on the ring of symmetric functions and function ν_2 to weight the last brick can find a generating function involving the statistic $\mathrm{fi}(\sigma)$. Let f_2 be the function defined such that $f_2(0)=1$ and $f_2(n)=(-x)(y-x)^{n-1}$. This is the function f_1 with the roles of x and y interchanged. Define ξ^{f_2} be the homomorphism with the property that

$$\xi^{f_2}(e_n)=\frac{(-1)^n}{n!}f_2(n)=\frac{(-1)^n}{n!}(-x)(y-x)^{n-1}.$$

Let ν_2 be the function to weight the last brick in a brick tabloid defined by

$$\nu_2(n)=\begin{cases}0 & \text{if } n<j,\\ \frac{y}{x}\left(\frac{y}{y-x}\right)^{j-1} & \text{if } n=j, \text{ and}\\ \frac{y}{x}\left(\frac{-x}{y-x}\right)\left(\frac{y}{y-x}\right)^{j-1} & \text{if } n>j.\end{cases}$$

THEOREM 1.29.

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n}x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)}z^{\mathrm{fi}(\sigma)}=\left(\frac{x-y}{x-y\mathbf{e}^{t(x-y)}}\right)\left(\frac{x-y(1-z)\mathbf{e}^{t(x-y(1-z))}}{x-y(1-z)}\right).$$

PROOF. To begin, we apply the homomorphism ξ^{f_2} to $n!p_{n,\nu_2}$. We have

$$\begin{aligned}n!\xi^{f_2}(p_{n,\nu_2}) &= n!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}w_{\nu_2}(B_{\lambda,(n)})\prod_{i=1}^{\ell(\lambda)}\xi^{f_2}(e_{\lambda_i})\\ &= \sum_{\lambda\vdash n}\binom{n}{\lambda}w_{\nu_2}(B_{\lambda,(n)})(-1)^{\ell(\lambda)}f_2(\lambda_1)\cdots f_2(\lambda_\ell).\end{aligned}\tag{1.32}$$

The weight ν_2 assures us that the last brick in $T\in B_{\lambda,(n)}$ must be at least j cells in length; otherwise the weight on such a T would be 0. Start to count (1.32) with $T\in B_{\lambda,(n)}$ where the last brick must be at least j cells in length. Use the binomial coefficient in (1.32) to fill the cells of T with a permutation of n such that each brick contains a *increasing* sequence. This is the reverse labeling as those used in many of the previous proofs. Use the $(-1)^{\ell(\lambda)} f_2(\lambda_1)\cdots f_2(\lambda_\ell)$ term to weight the bricks such that either an y or $-x$ appears in every nonterminal cell and x appears in the terminal cell of every brick. This, too, is different than the labeling in previous proofs.

Suppose the last brick in T is of length j. The weight on the last brick given in the definition of ν_2 enables us to replace the terminal x with y and replace the $j-1$ choices of either y or $-x$ with y. If the last brick in T is longer than j cells, the weight on the last brick enables us to replace the terminal x with y and replace the last j choices for y or $-x$ with one $-x$ followed by $j-1$ y's. One object which may be formed in this manner when $j=4$ is found below.

y	x	x	$-x$	y	x	$-x$	$-x$	y	y	y	y
3	5	1	4	6	9	2	7	8	10	11	12

The weighted sum over all possible objects formed in this way is equal to (1.32). Apply a brick breaking/combining involution where bricks are scanned from left to right looking for the first $-x$ or two consecutive bricks with an increase between them. If a $-x$ is scanned, break the brick into two and change the $-x$ to x. If an increase between two bricks is found, change the x in the middle of the new brick to $-x$. This is sign-reversing and weight-preserving. The fixed points under this involution correspond to permutations σ written in one line notation with $\mathrm{fi}(\sigma)=j$. The powers of x and y on a fixed point register descents and rises, respectively. Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)}z^{\mathrm{fi}(\sigma)} &= 1+\sum_{j=1}^{\infty} z^j\xi^{f_2}\left(\sum_{n=1}^{\infty} t^n p_{n,\nu_2}\right)\\ &= 1+\sum_{j=1}^{\infty} z^j\xi^{f_2}\left(\frac{\sum_{n=1}^{\infty}(-1)^{n-1}\nu_2(n)e_n t^n}{1+\sum_{n=1}^{\infty} e_n(-t)^n}\right).\end{aligned}$$

By the definition of ν_2 and ξ^{f_2}, this is equal to

$$1+\sum_{j=1}^{\infty} z^j\,\frac{\frac{t^j y^k}{j!}(y-x)+\frac{-xy^j}{(y-x)^j}\sum_{n=j+1}^{\infty}\frac{t^n(y-x)^n}{n!}}{(y-x)-x\sum_{n=1}^{\infty}\frac{t^n(y-x)^n}{n!}},$$

which may be rewritten to look like the statement of the theorem. □

Now, we may use the functions ν_1 and ν_2 which assigned weights to the final brick in a brick tabloid in conjunction with other homomorphisms we have used in this section. Specifically, if we wanted to know the generating function registering the final decreasing common descents within a collection of permutations, we could use the last brick weight ν_1 with the homomorphism $\xi_2^{f_1}$. To provide another example, we could use the function ν_1 together with the homomorphism $\xi_3^{f_1}$ to find that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} z^{\mathrm{fd}(\sigma)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}$$

$$= \left(\frac{x-y}{x - y\mathbf{e}_{p,q}^{t(x-y)}} \right) \left(1 + (1-z) \frac{y(\mathbf{e}^{xzt} - \mathbf{e}_{p,q}^{t(x-y)})}{x-y-xz} \right).$$

It is even possible to combine the functions ν_1 and ν_2 to alter our results involving major index statistics. Since the proof of our result for the major index statistic was a little different than other proofs included in this section and to further illustrate the machinery we are developing, a proof for the generating function which results from combining ν_1 and $\xi_4^{f_1}$ is given in Theorem 1.30 below.

THEOREM 1.30.

$$\sum_{n=0}^{\infty} \frac{t^n}{(u,v;q,p)_{n+1}} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma^{-1})} y^{\mathrm{ris}(\sigma^{-1})} z^{\mathrm{fd}(\sigma^{-1})} u^{\mathrm{des}(\sigma)} v^{\mathrm{ris}(\sigma)} q^{\mathrm{maj}(\sigma)} p^{\mathrm{comaj}(\sigma)}$$

$$= 1 + \sum_{m=1}^{\infty} z^m \left(\sum_{j=0}^{\infty} \frac{y x^{m-1} t^m q^{-mj} \begin{bmatrix} j+m \\ m \end{bmatrix}_{p,q}}{1 + \sum_{n=1}^{\infty} t^n (-y)(x-y)^{n-1} q^{-nj} \begin{bmatrix} j+n \\ n \end{bmatrix}_{p,q}} \right.$$

$$\left. - \sum_{j=0}^{\infty} \frac{y^2 \sum_{n=m+1}^{\infty} \frac{x^{j-1}}{(x-y)^j} (-y)(x-y)^{n-1} t^n q^{-nj} \begin{bmatrix} j+n \\ n \end{bmatrix}_{p,q}}{1 + \sum_{n=1}^{\infty} t^n (-y)(x-y)^{n-1} q^{-nj} \begin{bmatrix} j+n \\ n \end{bmatrix}_{p,q}} \right).$$

PROOF. Since both the homomorphism $\xi_4^{f_1}$ and the function ν_1 depend on the integer j, we will let m take the role of j in the definition of ν_1 throughout the rest of this proof. By applying $\xi_4^{f_1}$ to p_{n,ν_1}, brick tabloids like those in the proof of Theorem 1.26 may be constructed with the caveat that the final brick in a brick tabloid must have length at least m cells. Further, the x and y labeling of the final brick in the brick tabloid must match that found in the proof of Theorem 1.28.

Applying the involution found in the proof of Theorem 1.26 to this collection of objects, we find brick tabloids with positive sign, strict decreases in the words between bricks, a power of x for every weak increase, a power of y for every strict decrease, and final brick of length precisely m cells. The example of a fixed point when $j = 3$ and $m = 2$ may be found below.

x	x	x	y	x	y	x	x	x	y	x	y
$(p/q)^0$	$(p/q)^1$	$(p/q)^2$	$(p/q)^3$	$(p/q)^0$	$(p/q)^1$	$(p/q)^0$	$(p/q)^0$	$(p/q)^2$	$(p/q)^3$	$(p/q)^1$	$(p/q)^3$
0	1	2	3	0	1	0	0	2	3	1	3

Counting techniques found in the proof of Theorem 1.26 follow through in the same manner to show that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{t^n}{(u,v;q,p)_{n+1}} \frac{u^2 q^n}{v p^n} & \sum_{\substack{\sigma\in S_n \\ \mathrm{fd}(\sigma^{-1})=m}} x^{\mathrm{des}(\sigma^{-1})} y^{\mathrm{ris}(\sigma^{-1})} u^{\mathrm{des}(\sigma)} v^{\mathrm{ris}(\sigma)} q^{\mathrm{maj}(\sigma)} p^{\mathrm{comaj}(\sigma)} \\ &= \sum_{j=0}^{\infty} (v/u)^j \xi_4^{f_1} \left(\sum_{n=0}^{\infty} p_{n,\nu} t^n \right)\end{aligned}$$

which, following the definition of $\xi_4^{f_1}$ and ν_1 on (1.12), gives the statement of the theorem. □

In this section, we have just shown how to generalize and refine the methods introduced in Section 1.3 to produce many new results concerning generating functions counting permutation statistics over the symmetric group. In the next chapter we will continue in the same vein and derive many more results about permutations in the symmetric group.

CHAPTER 2

Generating functions for permutations

In this chapter we extend ideas found in Chapter 1 to find permutation statistics for various sets of permutations. Various generating functions will be found—some will reprove or generalize known results and some are completely new.

The chapter begins with Section 2.1 which contains results on the alternating permutations. In Section 2.2, we give methods which enable us to control the appearance of consecutive descents in permutations. Then, in Section 2.3, we describe how the number of both nonoverlapping and overlapping patterns appearing in permutations may be counted. Finally, in Section 2.4 we reprove and then generalize a well known result of Garsia and Gessel which gives a generating function for the triumvirate distribution of descents, inversions, and the major index statistics.

2.1. Alternating permutations

A permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is alternating provided $\sigma_i < \sigma_{i+1}$ if and only if i is even. Mnemonically, an alternating permutation may be interlaced with $<$ and $>$ signs:

$$\sigma_1 \ > \ \sigma_2 \ < \ \sigma_3 \ > \ \sigma_4 \ < \ \cdots \ \sigma_n.$$

An alternating permutation of an even number is called even alternating while an alternating permutation of an odd number is called odd alternating.

Define f_3 as the function on the nonnegative integers such that

$$f_3(n) = \begin{cases} 0 & \text{if } n \text{ is odd, and} \\ (-1)^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Let ξ^{f_3} be the homomorphism on Λ defined by

$$\xi^{f_3}(e_n) = \frac{(-1)^n}{n!} f_3(n).$$

Before use ξ^{f_3} to find generating functions for alternating permutations using the same techniques as exhibited in Section 1.5, we make some remarks about the overall methods we are developing. We have defined multiple homomorphisms on the ring of symmetric functions which involved functions f_1 and f_2. In this section, we will define multiple homomorphisms involving the functions f_3. These functions will be called weighting functions since they give the weightings of bricks in weighted brick tabloids. A listing of these functions is given in an Appendix. There is also an appendix containing the homomorphisms defined in this monograph.

THEOREM 2.1.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left|\{\sigma \in S_n : \sigma \textit{ is even alternating}\}\right| = \sec(t).$$

PROOF. By applying ξ^{f_3} to $n!h_n$, we have

$$n!\xi^{f_3}(h_n) = n!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,(n)}\prod_{i=1}^{\ell(\lambda)}\frac{(-1)^{\lambda_i}}{\lambda_i!}f_3(\lambda_i)$$
$$= \sum_{\lambda\vdash n}\binom{n}{\lambda}(-1)^{\ell(\lambda)}B_{\lambda,(n)}f_3(\lambda_1)\cdots f_3(\lambda_\ell).$$

This equation, together with the definition of f_3, supplies brick tabloids such that each brick must have even length and are interlaced with factors -1 in every other cell. Place decreasing sequences of integers in each brick and change the -1 on the last cell to 1. In this way, the weighted sum over all possible such decorated brick tabloids is equal to the above equation. One example of such a brick tabloid may be found below.

	-1		1		1		-1		-1		1
10	7	4	2	3	1	12	11	9	8	6	5

Scan the bricks from left to right looking for either a -1 or two consecutive bricks with a decrease between them. If a -1 is scanned, break the brick into two and change the -1 to 1. If two consecutive bricks with a decrease between them is scanned, reverse the operation. This is a sign-reversing involution with fixed points corresponding to the even alternating permutations of n. Equation (1.4) implies

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\left|\{\sigma\in S_n : \sigma \text{ is even alternating}\}\right| = \left(\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}\right)^{-1} = \sec(t). \qquad \square$$

THEOREM 2.2.
$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\left|\{\sigma\in S_n : \sigma \textit{ is odd alternating}\}\right| = \tan(t).$$

PROOF. First we will show $(n-1)!\xi^{f_3}(p_n)$ is the number of odd alternating permutations of $n-1$, then we will conclude with the statement of the theorem. We have

$$(n-1)!\xi^{f_3}(p_n) = (n-1)!\sum_{\substack{\lambda\vdash n \text{ has}\\ \text{even parts}}}\frac{1}{\lambda_1!\cdots\lambda_\ell!}w_n(B_{\lambda,(n)})(-1)^{\ell(\lambda)+\lambda_1/2+\cdots+\lambda_\ell/2}$$
$$= \sum_{\lambda\vdash n}\sum_{\substack{T\in B_{\lambda,(n)} \text{ has}\\ \text{even bricks } b_1,\ldots,b_k}}\frac{(n-1)!}{b_1!\cdots b_k!}b_k(-1)^{k+b_1/2+\cdots+b_k/2},$$

which simplifies to

$$\sum_{\lambda\vdash n}\sum_{\substack{T\in B_{\lambda,(n)} \text{ has}\\ \text{even bricks } b_1,\ldots,b_k}}\binom{n-1}{b_1,\ldots,b_{k-1},b_k-1}(-1)^{k+b_1/2+\cdots+b_k/2}. \tag{2.1}$$

Similar combinatorial objects as found in the proof of Theorem 2.1 may be constructed to count (2.1). Starting with $T\in B_{\lambda,(n)}$, force the bricks to be an even length, place -1 in every other cell in a brick, and a 1 in every terminal cell. The multinomial coefficient places a permutation of $n-1$ in the first $n-1$ cells such that each brick contains a decreasing sequence.

By applying the usual brick/breaking and combining involution, we are left with fixed points where every brick has length 2, there are increases between bricks, and only the first $n-1$ integers are permuted in the first $n-1$ cells. We provide an example below.

	1		1		1		1		1		1
9	4	6	2	3	1	11	8	10	5	7	

Fixed points correspond to odd alternating permutations, summing them proves $(n-1)!\xi^{f_3}(p_n)$ is the number of odd alternating permutations of $n-1$. We now have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}\left|\{\sigma\in S_n:\sigma \text{ is odd alternating}\}\right| &= \frac{1}{t}\xi^{f_3}\left(\sum_{n=1}^{\infty}p_nt^n\right)\\ &= \frac{1}{t}\frac{\sum_{n=1}^{\infty}(-1)^{n-1}n\xi^{f_3}(e_n)t^n}{\sum_{n=0}^{\infty}\xi^{f_3}(e_n)(-t)^n}\\ &= \frac{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^{2n-1}}{(2n-1)!}}{\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!}},\end{aligned}$$

which simplifies to $\tan(t)$ and completes the proof of the theorem. $\square$

Similar modifications to the homomorphism ξ_1^f in Section 1.5 follow through in the same way for the homomorphism ξ^{f_3} in this section. Specifically, let us define four new homomorphisms on the ring of symmetric functions such that

$$\begin{aligned}\xi_2^{f_3}(e_n) &= \frac{(-1)^n}{(n!)^m}f_3(n),\\ \xi_3^{f_3}(e_n) &= \frac{(-1)^n}{[n]_{p,q}!}q^{\binom{n}{2}}f_3(n),\\ \xi_4^{f_3}(e_n) &= (-1)^nq^{-nj}\begin{bmatrix}j+n\\ n\end{bmatrix}_{p,q}f_3(n), \qquad \text{and}\\ \xi_5^{f_3}(e_n) &= (-1)^np^{\binom{n}{2}-nj}q^{\binom{n}{2}}\begin{bmatrix}j+1\\ n\end{bmatrix}_{p,q}f_3(n)\end{aligned}$$

The homomorphism $\xi_2^{f_3}$ may be used to find a generating function restrict multiples of permutations by an alternating version of common descents. That is, using $\xi_2^{f_3}$, we may find that

$$\begin{aligned}(2.2)\qquad \sum_{n=0}^{\infty}\frac{t^n}{(n!)^m}&\left|\{\sigma^1,\dots,\sigma^m\in S_n:\text{common descents even alternate}\}\right|\\ &= \frac{1}{\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{((2n!))^m}}.\end{aligned}$$

by applying $\xi_2^{f_3}$ to h_n and using the labeling scheme for brick tabloids introduced in the proof of Theorem 1.23.

To find the analogous generating function to (2.2) for the number of odd alternating m-tuples of permutations, we must apply $\xi_2^{f_3}$ to a symmetric function other than h_n or p_n. We must apply $\xi_2^{f_3}$ to the symmetric function p_{n,n^m} so that the last brick in a brick tabloid is weighted with n^m rather than simply n as in the proof of Theorem 2.2. This would change the decorated brick tabloids to have m rows of permutations of $m-1$. Then we would find through the methods in Theorem 2.2 that

$$\sum_{n=0}^{\infty} \frac{t^n}{(n!)^m} \left|\{\sigma^1, \ldots, \sigma^m \in S_n : \text{common descents odd alternate}\}\right| = \frac{\sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^{2n-1}}{((2n-1)!)^m}}{\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{((2n!))^m}}.$$

Using $\xi_3^{f_3}$, we may show without much more effort than in the proofs of Theorem 2.1 and Theorem 2.2 that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \text{ is} \\ \text{alternating}}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} = \frac{1 + \sin_{p,q}(t)}{\cos_{p,q}(t)}. \tag{2.3}$$

The justification for (2.3) comes from the p, q-labeling found in the proof of by "p, q-analoging" the proof of Theorem 2.2 with the help of the homomorphism which sends e_n to $(-1)^{n/2} q^{\binom{n}{2}}/[n]_{p,q}!$ and the last brick weight $[n]_{p,q} q^{1-n}$.

The weighting function $\xi_4^{f_3}$ creates brick tabloids where each brick must have length 2. Tracing back the proof of Theorem 1.26, this would force the permutation τ_r^{-1} to be even alternating. Applying the same calculations as in the above proof, this would give the result

$$\sum_{n=0}^{\infty} \frac{t^n}{(u,v;q,p)_{n+1}} \sum_{\substack{\sigma \in S_n \\ \sigma^{-1} \text{ is even alt.}}} u^{\mathrm{des}(\sigma)} v^{\mathrm{ris}(\sigma)} q^{\mathrm{maj}(\sigma)} p^{\mathrm{comaj}(\sigma)} = \sum_{j=0}^{\infty} \frac{v^{j+1}}{u^{j+2}} \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} q^{-2nj} \begin{bmatrix} j+2n \\ 2n \end{bmatrix}_{p,q} \right)^{-1}.$$

Thus we have refined the even alternating permutations by descents, rises, major index, and the co-major index of the inverse. Odd alternating permutations may also be refined in a similar manner. Note that the inverse permutations are restricted by the weighting function f_3. In addition, through the relabeling ideas indicated immediately after the proof of Theorem 1.26, the major index and co-major index may be changed to the right-to-left major index and the co-right-to-left major index with slight adjustments to the powers of p and q in the generating function.

Our methods can easily find generating functions for generalizations of alternating permutations. For a positive integer j, a j-alternating permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has the property that $\sigma_i < \sigma_{i+1}$ if and only if j divides i. In this way, 2-alternating permutations are simply the alternating permutations.

Let ξ_6 be the homomorphism defined on e_n such that $\xi_6(n) = 1/(nj)!$. For $0 \leq k < n$, let

$$\nu_3(n) = \frac{(nj)!}{(nj-k)!}$$

In the same way as ξ^{f_3} and the last brick weighting combined in the proof of Theorem 2.2, ξ_6 and ν_3 will combine to give the proof of Theorem 2.3 below.

THEOREM 2.3. *For $j \geq 1$ and $0 \leq k < j$,*

$$\sum_{n=1}^{\infty} \frac{t^{nj-k}}{(nj-k)!} \left|\{\sigma \in S_{nj-k} : \sigma \text{ is } j\text{-alternating}\}\right| = \frac{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^{nj-k}}{(nj-k)!}}{\sum_{n=0}^{\infty}(-1)^{n}\frac{t^{nj}}{(nj)!}}$$

PROOF. Let us first expand $(nj-k)!\xi_6(p_{n,\nu_3})$ using Theorem 1.15. We have

$$\begin{aligned}
(nj-k)!\xi_6(p_{n,\nu_3}) &= (nj-k)!\sum_{\lambda \vdash n} \frac{1}{(j\lambda_1)!\cdots(j\lambda_\ell)!} w_{\nu_3}(B_{\lambda,(n)})(-1)^{n-\ell(\lambda)} \\
&= \sum_{\lambda \vdash n} \sum_{\substack{T \in B_{\lambda,(n)} \text{ has} \\ \text{bricks } b_1,\ldots,b_m}} \frac{(nj-k)!}{(b_1 j)!\cdots(b_m j)!} \frac{(b_m j)!}{(b_m j-k)!}(-1)^{n-m} \\
&= \sum_{\lambda \vdash n} \sum_{\substack{T \in B_{\lambda,(n)} \text{ has} \\ \text{bricks } b_1,\ldots,b_m}} \binom{nj-k}{b_1 j,\ldots,b_{m-1}j, b_m j-k}(-1)^{n-m}. \qquad (2.4)
\end{aligned}$$

Similar combinatorial objects as found in the proof of Theorem 2.2 may be constructed to count (2.4). Take $T \in B_{\lambda,(n)}$ and multiply the length of every brick by j. Place -1 in every j^{th} cell in a brick and a 1 in every terminal cell. The multinomial coefficient in (2.4) allows for the first $nj-k$ cells to be filled with a permutation of $nj-k$ such that bricks contain decreasing sequences. An example of such an object when $n = 3, j = 4$, and $k = 3$ is found below.

			-1				1				1
9	8	7	6	4	3	2	1	5			

By applying the same involution as that applied to objects like that in the proof of Theorem 2.2, we are left with fixed points where every brick has length j, there are increases between bricks, and only the first $nj-k$ integers are permuted in the first $nj-k$ cells. Fixed points correspond to j-alternating permutations, summing them gives

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{t^{nj-k}}{(nj-k)!} \left|\{\sigma \in S_{nj-k} : \sigma \text{ is } j\text{-alternating}\}\right| &= \frac{1}{t^k}\xi_6\left(\sum_{n=1}^{\infty} p_{n,\nu_3} t^{nj}\right) \\
&= \frac{1}{t^k}\frac{\sum_{n=1}^{\infty}(-1)^{n-1}\nu_3(n)\xi_6(e_n)t^{nj}}{\sum_{n=0}^{\infty}(-1)^n\xi_6(e_n)t^{nj}} \\
&= \frac{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^{nj-k}}{(nj-k)!}}{\sum_{n=0}^{\infty}(-1)^{n}\frac{t^{nj}}{(nj)!}}
\end{aligned}$$

This completes the proof. □

Summing the generating functions in Theorem 2.3 for $0 \leq k < j$, succinct expressions involving elementary functions can count these classes of permutations. For example, taking $j = 2$, the generating function for alternating permutations starting at $n = 1$ is equal to $\sec(t) + \tan(t) - 1$. Similarly, the generating function for 3-alternating permutations is

$$\frac{3 + 2\sqrt{3}\mathbf{e}^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)}{\mathbf{e}^{-t} + 2\mathbf{e}^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)},$$

and the generating function for 4-alternating permutations is

$$\frac{1 + \sqrt{2}\sin\left(\frac{t}{\sqrt{2}}\right)\cosh\left(\frac{t}{\sqrt{2}}\right) + \sin\left(\frac{t}{\sqrt{2}}\right)\sinh\left(\frac{t}{\sqrt{2}}\right)}{\cos\left(\frac{t}{\sqrt{2}}\right)\cosh\left(\frac{t}{\sqrt{2}}\right)}.$$

We may refine subsets of m-tuples of permutations so that we count those collections with j-alternating descents. Further, the above equations may be refined by powers of p and q to register inversions and co-inversions over the j-alternating permutations and also may be refined by major index statistics in the same spirit as the alternating permutations.

There have been various uses and extensions of alternating permutations. Euler correctly gave the expansion of $\sec(t)$ up to t^{16} (his coefficient of t^{18} is erroneous) and their connection to permutations [**33**]. This work probably spurred Sylvester to call the number of alternating permutations Euler numbers [**68**]. In the literature, they are also referred to as the number of up-down permutations and zig zag sequences.

André's papers in 1879 and 1881 are credited as containing the first proof that the generating function for the alternating permutations is $\sec(t) + \tan(t)$ [**1, 2**]. His ideas were included in Netto's 1927 combinatorics text [**57**]. After Entringer reproved André's result in 1966 [**31**], his work was recounted and reworked in a series of papers by Carlitz and by Carlitz and Scoville over the next decade [**20, 25, 21, 22**]. The technique of proofs in these works was to use recursions given by the definition of alternating permutations to find a differential equation for the generating function.

The q-analogue in (2.3) and the generalization of alternating permutations found in Theorem 2.3 was discovered by Stanley by working with binomial posets [**65**].

2.2. Consecutive descents

Let f_4 be a weighting function such that $f_4(0) = 1$ and for all $n \geq 1$,

$$f_4(n) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^i (-y)^{n-i}$$

for indeterminates x and y. A permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a 2-descent if there is an index i such that $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$ [**43**]. There is a relationship between the weighting function f_4 and permutations without 2-descents.

THEOREM 2.4.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \text{ without} \\ \text{2-descents}}} x^{\operatorname{des}(\sigma)} y^{\operatorname{ris}(\sigma)} = \frac{\mathbf{e}^{ty/2}}{\cos\left(\frac{t\sqrt{4xy-y^2}}{2}\right) - \frac{y}{\sqrt{4xy-y^2}}\sin\left(\frac{t\sqrt{4xy-y^2}}{2}\right)}.$$

Proof. First, let us show that $\binom{n-i}{i}$ counts the number of rearrangements of i "x"'s and $n-i-1$ "$-y$"'s where no two consecutive x's appear. For example, one such rearrangement when $i=5$ and $n=17$ is

$$-y\ -y\ x\ -y\ -y\ x\ -y\ -y\ x\ -y\ -y\ x\ -y\ x\ -y\ -y$$

Denote the first x and then subsequent pairs $-y\ x$ as $|$ and denote the rest of the $-y$'s as $\star$. In this way, the rearrangement above becomes

$$\star\ \star\ |\ \star\ |\ \star\ |\ \star\ |\ |\ \star\ \star.$$

Any rearrangement of i bars and $n-2i$ stars corresponds to a rearrangement of i "x"'s and $n-i-1$ "$-y$"'s where no two consecutive x's appear. There are $\binom{n-i}{i}$ bar and star configurations, thus the binomial coefficient counts the desired objects.

The set $\mathcal{T}_{f_4}$ may now be described. Use the subset of the first n positive integers assigned to each brick to fill bricks with decreasing sequences. The function f_4 allows for the first $n-1$ cells in a brick of length n to be labeled with x's and $-y$'s such that no two consecutive x's appear. The factor of -1 and the weighting function f_4 label the last cell in each brick with y. Below we give an example of one such $T \in \mathcal{T}_{f_4}$.

$-y$	x	y	y	x	$-y$	y	x	$-y$	x	$-y$	y
11	10	8	1	9	5	3	12	7	6	4	2

Since each brick ends with y and there are no two consecutive x's within each brick, there can never be any two consecutive x's in all of T.

We can apply the sign-reversing weight-preserving involution I_ξ found in Section 1.3 to $\mathcal{T}_{f_4}$. That is, scan the bricks from left to right for the first instance of either $-y$ or a decrease between two bricks. If the occurrence is a $-y$, break the brick into two and change the sign on the $-y$ to a y. If a decrease between two bricks occurs first, apply the reverse operation. The set of fixed points we are left with are those $T \in \mathcal{T}_{f_4}$ where no brick is longer than 2 cells, there are no decreases between bricks, and there are no $-y$ labels. These fixed points correspond to elements in S_n without 2-descents and the powers of x and y are appropriately dispersed to register descents and rises. An example of such a fixed point may be found below.

x	y	y	y	x	y	y	x	y	x	y	y
11	5	9	10	12	1	2	6	4	7	1	3

We have

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\substack{\sigma\in S_n \text{ without}\\ \text{2-descents}}} x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)} &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\substack{T\in\mathcal{T}_{f_4}\\ I_\xi(T)=T}} w(T)\\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{T\in\mathcal{T}_{f_4}} w(T)\\
&= \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}x^i(-y)^{n-i}\right)^{-1}. \qquad (2.5)
\end{aligned}$$

At this point we have a generating function for permutations in S_n without 2-descents refined by descents and rises. To complete the proof of this theorem, we will indicate how (2.5) can be written in the fancy manner as in the statement of the theorem. The study of Chebyshev polynomials of the second kind [**63**] gives

$$\frac{\left(t+\sqrt{t^2-1}\right)^{n+1}-\left(t-\sqrt{t^2-1}\right)^{n+1}}{2\sqrt{t^2-1}} = \sum_{i=0}^{\lfloor n/2\rfloor}\binom{n-i}{i}(-1)^i(2t)^{n-2i}, \qquad (2.6)$$

so taking $t=\sqrt{y/(4x)}$ in the above equation, (2.5) may be rewritten to look like

$$\left(\sum_{n=0}^{\infty}\frac{(-yt)^n x^{\frac{n}{2}}}{2n!y^{\frac{n}{2}}\sqrt{\frac{y}{4x}-1}}\left(\left(\sqrt{\frac{y}{4x}}+\sqrt{\frac{y}{4x}-1}\right)^{n+1}-\left(\sqrt{\frac{y}{4x}}-\sqrt{\frac{y}{4x}-1}\right)^{n+1}\right)\right)^{-1}$$

which in turn may be simplified to the desired expression. □

There is a straightforward generalization of permutations in S_n without 2-descents. A permutation $\sigma=\sigma_1\cdots\sigma_n\in S_n$ has a j-descent if there is an index i such that $\sigma_i>\sigma_{i+1}>\cdots>\sigma_{i+j}$. Notice that in a j-descent there are $j+1$ integers and j "$>$" signs. To find a generating function for permutations of n without j-descents, we can define a weighting function which will weight bricks such that j consecutive x labels never occur. To this end, let $R_{n,i,j}$ be the number of rearrangements of i x's and $n-i$ $(-y)$'s without j consecutive x's. The number $R_{n,i,j}$ can be found by an assortment of different methods, but we will not explicitly need such a expression—especially a messy one. Let f_5 be the weighting function in the indeterminates x and y such that for $n\geq 1$,

$$f_5(n)=\sum_{i=0}^{\infty}R_{n-1,i,j}x^i(-y)^{n-i}.$$

This is equal to f_4 when $j=2$.

Theorem 2.5. *For $j\geq 2$,*

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\substack{\sigma\in S_n \text{ without}\\ j\text{-descents}}} x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)} = \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{i=0}^{\infty}R_{n-1,i,j}x^i(-y)^{n-i}\right)^{-1}.$$

Proof. The set $\mathcal{T}_{f_5}$ contains brick tabloids filled with decreasing sequences, x and $-y$ labels such that there are never j consecutive x's, and bricks which end in y. Employing the brick breaking/combining involution found in the proof of Theorem 2.4, we are left with fixed points which correspond to permutations without j-descents. This completes the proof. $\square$

We have given a simplification of the generating function in Theorem 2.5 in terms of elementary functions in Theorem 2.4 when $j = 2$, but such a simplification is difficult to find in the general case. However, there is a nice specialization of Theorem 2.5 in the case of taking $x = y = 1$. In this instance, we are simply counting the number of permutations without j-descents. Before we state this specialization we first prove Lemma 2.6 below.

LEMMA 2.6.

$$\sum_{i=0}^{\infty}(-1)^i R_{n,i,j} = \begin{cases} (-1)^n & \textit{if } j+1 \textit{ divides } n, \\ (-1)^{n+1} & \textit{if } j+1 \textit{ divides } n+1, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. Let $r = r_1 \cdots r_n \in R_{n,i,j}$. For convenience and clarity, we will write r as a sequence of n 0's and 1's instead of x's and $(-y)$'s. Define the sign of r as $(-1)^i$, where i is the number of 0's in r. To prove Lemma 2.6, a sign-reversing involution I will be given.

If $r_1 = 0$, let $I(r)$ be equal to r with r_1 changed to 1. If $r_1 = 1$ and $0\, r_2 \cdots r_n$ does not contain j consecutive 0's, let $I(r)$ be equal to r with r_1 changed to 0. If $r_1 = 1$ and $0\, r_2 \ldots r_n$ does contain j consecutive 0's, then either

$$r = 1\ \underbrace{0\ \cdots\ 0}_{j-1} \qquad \text{or}$$

$$r = 1\ \underbrace{0\ \cdots\ 0}_{j-1}\ 1\ r_{j+2}\ \cdots\ r_n.$$

In the former case, let $I(r) = r$. In the latter case, proceed inductively by the above description of the involution I on $r_{j+2}\ \cdots\ r_n$ to find $I(r)$.

We have defined I as an involution which pairs every non-fixed r with a rearrangement $I(r)$ of opposite sign. There are only two possible fixed points:

$$r = \underbrace{1\ 0\ \cdots\ 0\ 1}_{j+1}\ \underbrace{1\ 0\ \cdots\ 0\ 1}_{j+1}\ \cdots\ \underbrace{1\ 0\ \cdots\ 0\ 1}_{j+1} \qquad \text{and}$$

$$r = \underbrace{1\ 0\ \cdots\ 0\ 1}_{j+1}\ \underbrace{1\ 0\ \cdots\ 0\ 1}_{j+1}\ \cdots\ \underbrace{1\ 0\ \cdots\ 0}_{j}.$$

In the first case, $(j+1)$ divides n and the sign is $(-1)^{\frac{n}{j+1}(j-1)} = (-1)^n$. In the second case, $j+1$ divides $n+1$ and the sign is $(-1)^{\frac{n+1}{j+1}(j-1)} = (-1)^{n+1}$. This completes the proof. $\square$

Using this lemma, the simple generating function in Corollary 2.7 may be stated.

COROLLARY 2.7. *For $j \geq 1$,*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, |\{\sigma \in S_n : \sigma \textit{ is without } j\textit{-descents } \}| = \frac{j+1}{(1-\alpha^j)\,\mathbf{e}^{\alpha t} + \cdots + (1-\alpha)\mathbf{e}^{\alpha^j t}}$$

where $\alpha = \mathbf{e}^{2\pi i/(j+1)}$ is a $j+1^{\text{st}}$ root of unity.

Proof. Taking $x = y = 1$ in Theorem 2.5,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left|\{\sigma \in S_n : \sigma \text{ is without } j\text{-descents }\}\right|$$
$$= \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{i=0} (-1)^{n-i} R_{n-1,i,j} \right)^{-1}$$
$$= \left(\sum_{n=0}^{\infty} \frac{t^{(j+1)n}}{((j+1)n)!} - \sum_{n=0}^{\infty} \frac{t^{(j+1)n+1}}{((j+1)n+1)!} \right)^{-1}$$

where the last step follows from Lemma 2.6. The above string of equalities may be simplified to read

$$\left(\frac{\mathbf{e}^t + \cdots + \mathbf{e}^{\alpha^j t}}{j+1} - \int \frac{\mathbf{e}^t + \cdots + \mathbf{e}^{\alpha^j t}}{j+1}\, dt \right)^{-1}$$

which is equal to the desired expression. $\square$

In the case of $j = 1$ in Corollary 2.7, we find the generating function for the number of permutations with no descents is $\mathbf{e}^t$ and taking $j = 2$ in Corollary 2.7, one may find the generating function in the statement of Theorem 2.4 when $x = y = 1$.

We will now start a discussion on how to find generating functions for permutations where the occurrences of rises must be spaced a predetermined distance apart. A permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has a rise at place i if $i = 0, i = n$, or $\sigma_i < \sigma_{i+1}$ for $i = 1, \ldots, n-1$. Define $\sigma \in S_n$ to be j-spaced if σ has rises spaced by at least j. For example, 4 3 1 6 5 2 is 3-spaced (as well as 2-spaced and 1-spaced).

Let j be some positive integer and f_6 be the weighting function in the variables x and y such that $f_6(0) = 1$ and for $n \geq 1$,

$$f_6(n) = \begin{cases} 0 & \text{if } n < j, \text{ and} \\ \displaystyle\sum_{i=0}^{\lfloor (n-j)/j \rfloor} \binom{n-j-(j-1)i}{i} x^{n-i-1}(-y)^{i+1} & \text{if } n \geq j. \end{cases}$$

The weighting function f_6 has been constructed to yield j-spaced permutations of n.

Theorem 2.8. *For $j \geq 1$,*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \text{ is} \\ j\text{-spaced}}} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)}$$
$$= \left(1 + \sum_{n=j}^{\infty} \frac{t^n}{n!} \sum_{i=0}^{\lfloor (n-j)/j \rfloor} \binom{n-j-(j-1)i}{i} x^{n-i-1}(-y)^{i+1} \right)^{-1}.$$

Proof. First we show that the binomial coefficient $\binom{n-j-(j-1)i}{i}$ counts the number of ways to rearrange i "$-y$"'s and $n-i-1-(j-1)$ "x"'s such that we begin with at least $j-1$ x's and there are at least $j-1$ x's between each $-y$. For example, one such rearrangement when $j = 3$, $n = 20$, and $i = 4$ is

$$x\ x\ \ -y\ x\ x\ \ -y\ x\ x\ x\ x\ \ -y\ x\ x\ x\ \ -y\ x\ x.$$

Consider each sequence of $(j-1)$ x's followed by $-y$ as $|$ and consider the remaining x's as $\star$. In this way, the rearrangement above becomes

$$| \; | \; \star \; \star \; | \; \star \; | \; \star \; \star.$$

Any such configuration of bars and stars will give an acceptable arrangement of x's and $-y$'s. There are i bars and $n-j-ji$ stars; thus there are $\binom{n-j-(j-1)i}{i}$ possible rearrangements. Thus, along with the decreasing sequences coming from the subset of the first n positive integers, the $T \in \mathcal{T}_{f_6}$ can be made to have each y or $-y$ label separated by at least $j-1$ x's. An example of one such object when $j=4$ in $\mathcal{T}_{f_6}$ is displayed below.

x	x	x	$-y$	x	x	x	y	x	x	x	y
12	10	8	7	6	5	4	3	11	9	2	1

The brick breaking/combining involution appearing in the previous two proofs leave fixed points corresponding to $\sigma \in S_n$ where σ is j-spaced. This completes the proof. $\square$

In general, the generating function in Theorem 2.8 does not simplify easily; but in the case of $j=2$ we may use the generating function for Chebyshev polynomials of the second kind in (2.6) and apply similar simplifications indicated in the proof of Theorem 2.4 to find

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma \text{ is 2-spaced}}} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} = \frac{\mathrm{e}^{-tx/2}}{\cos\left(\frac{t\sqrt{4yx-x^2}}{2}\right) - \frac{x}{\sqrt{4yx-x^2}} \sin\left(\frac{t\sqrt{4yx-x^2}}{2}\right)}.$$

We showed in the beginning of this section that we are able to force bricks in a fixed point to have length less than j by never weighting a brick with j consecutive x's. This enabled us to find a generating function for the number of permutations in the symmetric group without j consecutive descents. Then, we showed that by forcing each brick in a fixed point to have length longer than j we were able to space the rises in a permutation. There are, of course, many variations on these ideas. In the same vein as all of the weighting functions and theorems in this section, there are numerous different weighting functions which may be defined to restrict the lengths of bricks which may appear within a fixed point. Say, for example, that for a set of positive integers $\{k_1, k_2, \dots\}$ one wanted to find a generating function for the number of permutations in the symmetric group without precisely $k_1, k_2, \dots$ consecutive descents. Let $R_{i,n-i-1}$ be the number of rearrangements of i x's and $n-i-1$ y's such that there are never $k_1, k_2, \dots$ consecutive x's. We may define a weighting function f such that $f(0)=1$ and

$$f(n) = \sum_{i=0}^{n-1} R_{k,n-k-1} x^k (-y)^{n-k}.$$

Using the same weighting of brick tabloids and using the same brick breaking and combining involutions found in the proofs of Theorem 2.4, Theorem 2.5, and Theorem 2.8, it follows that the exponential generating function counting descents and rises over the subset of the symmetric group without precisely $k_1, k_2, \ldots$ consecutive descents is equal to

$$\left(1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{i=0}^{n-1} R_{k,n-k-1} x^k (-y)^{n-k}\right)^{-1}$$

For example, we may force a fixed point to never have a brick of length two. This would enable us to count the number of permutations in the symmetric group without precisely one consecutive descent. That is, define a weighting function f_7 such that $f_7(0) = 1$ and

$$f_7(n) = (-y)^n + \sum_{i=2}^{n-1} x^i (-y)^{n-i} \sum_{k=1}^{\min\{n-i, \lfloor i/2 \rfloor\}} \binom{n-i}{k} \binom{i-k-1}{k-1}$$

if $n \geq 1$. This weighting function will give a sequence to label a brick of length n with x's and $(-y)$'s such that there will never be precisely one consecutive x. That is, this weighting function gives rise to Theorem 2.9.

THEOREM 2.9.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \text{ never has} \\ \text{precisely one consec. des.}}} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)}$$
$$= \left(\mathbf{e}^{-yt} + \sum_{n=3}^{\infty} \frac{t^n}{n!} \sum_{i=2}^{n-1} x^i (-y)^{n-i} \sum_{k=1}^{\min\{n-i, \lfloor i/2 \rfloor\}} \binom{n-i}{k} \binom{i-k-1}{k-1}\right)^{-1}.$$

PROOF. By the above discussion and the involutions found in the proofs of the previous three theorems, it remains to be shown that the definition of the weighting function f_7 actually gives rise to rearrangements of x's and $-y$'s such that each rearrangement never has precisely one consecutive x and ends in $-y$. That is the same as verifying that

$$\sum_{k=1}^{\min\{n-i, \lfloor i/2 \rfloor\}} \binom{n-i}{k} \binom{i-k-1}{k-1} \tag{2.7}$$

is the number of rearrangements of i x's and $n-i-1$ $-y$'s such that precisely one x never appears consecutively. One such rearrangement may be found below when taking $n = 20$ and $i = 12$:

$$-y\ x\ x\ x\ -y\ -y\ -y\ x\ x\ -y\ x\ x\ x\ x\ x\ -y\ -y\ x\ x$$

Let us represent each x with a star and each $-y$ with a bar. The above rearrangement then looks like

$$| \star \star \star ||| \star \star | \star \star \star \star \star || \star \star. \tag{2.8}$$

First, choose exactly how many places stars will be written. In the above configuration, the stars were placed in four places. If stars will be written in k places, there are $\binom{n-i}{k}$ choices for where these k places may be found within the bars of a stars-and-bars configuration. Notice that this number k must fall between 1 and $n-i$. Further, since we must place a minimum of two stars in each of these k places, k must also be smaller than $\lfloor i/2 \rfloor$.

We must place at least two stars in each of the k choices for places. Thus, let us take a star and bar configuration like that found in (2.8) and change it into that below

$$\star \mid \mid \star \ \star \ \star \ \mid$$

by removing all of the extraneous bars and removing two stars from each of our choices of placements. These second star and bar configurations consist of $i-2k$ stars and $k-1$ bars and have no restrictions. There are $\binom{i-k-1}{k-1}$ ways to rearrange these bars and stars. Therefore, combined with the choice for k along with the binomial coefficient $\binom{n-i}{k}$, this proves that the number in (2.7) counts the desired objects.

The rest of this proof is the same as the proofs in Theorem 2.4 and Theorem 2.8 in that we weight the brick tabloids with the appropriate rearrangements of x's and $-y$'s, perform a brick breaking and combining involution. This completes the proof. □

These same techniques we have given thus far may be used to examine the set of permutations with descents appearing in blocks of length j. Define

$$S_{n,j} = \{\sigma \in S_{nj} : \text{if } j \text{ does not divide } i, \text{ then } \sigma_i > \sigma_{i+1}\}.$$

For $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let $\mathrm{jdes}(\sigma)$ be the statistic counting the number of indexes i such that $\sigma_i > \cdots > \sigma_{i+j}$; i.e., the number of times there are j consecutive descents. When $j=1$, jdes and des give the same count. Define $\mathrm{jris}(\sigma)$ to be the statistic counting the number of indexes i such that $\sigma_i > \cdots > \sigma_{i+j+1}$ is not true so that for any permutation of n, $\mathrm{jdes}(\sigma) + \mathrm{jris}(\sigma) = n$. When $j=1$, the statistic $\mathrm{jris}(\sigma) = \mathrm{ris}(\sigma)$ for any permutation in S_n.

One weighting function in the variables x^j and y^j which will help us keep track of jdes and jris over $S_{n,j}$ is f_8 which is defined such that for $n \geq 0$,

$$f_8(n) = \begin{cases} 0 & \text{if } j \text{ does not divide } n, \text{ and} \\ -y^j \left(x^j - y^j\right)^{\frac{n}{j}-1} & \text{if } j \text{ divides } n. \end{cases}$$

In the special case $j=1$, the weighting function f_8 is the same as f_1.

THEOREM 2.10. *For $j \geq 1$,*

$$\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!} \sum_{\sigma \in S_{n,j}} x^{\mathrm{jdes}(\sigma)} y^{\mathrm{jris}(\sigma)} = \frac{x^j - y^j}{x^j - y^j \sum_{n=0}^{\infty} \frac{(t^j(x^j-y^j))^n}{(jn)!}}.$$

PROOF. Elements in $\mathcal{T}_{f_8}$ consist of those brick tabloids where every brick has length a multiple of j, each block of j cells not terminating a brick is weighted with either x^j or $-y^j$, and the final j block of cells is labeled with y^j. An example of one such $T \in \mathcal{T}_{f_8}$ when $j=3$ is found below.

		x^3			y^3			$-y^3$			y^3
12	9	8	5	3	2	11	10	7	6	4	1

Apply the brick breaking/combining involution we have used in many of the preceding proofs on blocks of j cells. The fixed points correspond to permutations in $S_{n,j}$ where there the powers of x and y register the statistics jdes and jris, respectively. We have

$$\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!} \sum_{\sigma \in S_{n,j}} x^{\mathrm{jdes}(\sigma)} = \left(1 + \sum_{n=1}^{\infty} \frac{t^{jn}}{(jn)!} (-y^j)(x^j - y^j)^{n-1} \right)^{-1},$$

which in turn may be simplified to the desired expression. $\square$

The special case of Theorem 1.17 is revealed by taking $j = 1$ in Theorem 2.10. Succinct generating functions can be found for other small values of j; for example, taking $j = 2$ gives

$$\frac{x^2 - y^2}{x^2 - y^2 \cosh(t\sqrt{x^2 - y^2})},$$

and by taking $j = 4$,

$$\frac{2(x^4 - y^4)}{2x^4 - y^4 \left(\cos(t\sqrt[4]{x^4 - y^4}) + \cosh(t\sqrt[4]{x^4 - y^4}) \right)}.$$

Suppose we wanted to register the classic notion of descents over $S_{n,j}$. For all $\sigma \in S_{n,j}$, it is not difficult to show

$$\frac{\mathrm{jdes}(\sigma)}{j} + j(n-1) = \mathrm{des}(\sigma). \tag{2.9}$$

This follows from the fact that $S_{n,j}$ is the set of permutations where blocks of j must register descents. Therefore, if we wanted generating functions for descents over $S_{n,j}$, we only need to adjust the powers of x and t in the generating function in Theorem 2.10. However, we have another option: describing another weighting function. Let f_9 be the weighting function in the variables x and y such that for $j \geq 1$ and $n \geq 0$,

$$f_9(n) = \begin{cases} 0 & \text{if } j \text{ does not divide } n, \text{ and} \\ -x^{\frac{(j-1)n}{j}} y\,(x-y)^{\frac{n}{j}-1} & \text{if } j \text{ divides } n. \end{cases}$$

This will enable us to rid ourselves of the jdes statistic and concentrate on simply descents as Theorem 2.11 below shows.

THEOREM 2.11. *For $j \geq 1$,*

$$\sum_{n=0}^{\infty} \frac{t^{nj}}{(nj)!} \sum_{\sigma \in S_{n,j}} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} = \frac{x - y}{x - y \sum_{n=0}^{\infty} \frac{(x^{j-1} t^j (x-y))^n}{(jn)!}}.$$

PROOF. Elements of $\mathcal{T}_{f_9}$ are those brick tabloids where each brick must be a multiple of j and the the weight on each brick of length ℓ is $x^{\frac{(j-1)\ell}{j}} y\,(x + (-y))^{\frac{\ell}{j}-1}$. With this factor, label every j^{th} cell in a brick with either x or $-y$ and label the last cell in a brick with y. Then, label the rest of the cells with x. Apply the usual brick breaking/combining involution on this collection of objects. The figure below gives an example of a fixed point under this weight-preserving, sign-reversing involution when $j = 3$.

x	x	x	x	x	y	x	x	y	x	x	y
11	7	4	3	2	1	12	10	8	9	6	5

By viewing the integers as elements of $S_{n,j}$, the statement of the theorem follows by summing over all possible fixed points and writing down the corresponding generating function. $\square$

These same ideas may be applied to most other results we have found thus far. For a particularly nice example, we turn to the case of permutations without 2-descents. Recall from Section 2.1 that ξ^{f_4} is the homomorphism defined by

$$\xi^{f_4}(e_n) = \frac{(-1)^n}{n!} f_4(n) = \frac{(-1)^n}{n!} \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^i (-y)^{n-i}.$$

To pair with ξ^{f_4}, we define the function ν_4 to weight the last brick in a brick tabloid by

$$\nu_4(n) = \begin{cases} 1 & \text{if } n = 1, \text{ and} \\ (-y)\dfrac{f_4(n-1)}{f_4(n)} & \text{if } n \geq 2. \end{cases}$$

These two functions will team up to provide a generating function for those permutations without 2-descents and the property that $\sigma_{n-1} < \sigma_n$. This means that the last brick in a brick tabloid will have length 1. Since permutations without 2-descents come from brick tabloids with bricks of length either 1 or 2, we are not losing any generality by forcing the last brick to have length 1.

Theorem 2.12.

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \text{ without} \\ \text{2-descents and } \sigma_{n-1} < \sigma_n}} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} = \frac{2y}{\sqrt{4x-y^2}\cot\left(\frac{t\sqrt{4x-y^2}}{2}\right) - y}.$$

Proof. In the proof of Theorem 2.4 we showed that applying ξ^{f_4} to $n!h_n$ gave brick tabloids in which each brick contains a decreasing sequence and a weighting such that the final cell contains y and every other cell contains an x or $-y$ in such a way that no two x's appear. An example of such an object is shown on page 57. This weighting of x's and $-y$'s came directly from the definition of f_4. Specifically, the last brick of length b is weighted with $f_4(b)$.

When applying ξ^{f_4} to $n!p_{n,\nu_4}$, the same objects as described above may be formed with the additional weight of $\nu_4(b)$ assigned to the last brick of length b. From the definition of ν_4, the weight $f_4(b)$ is canceled and in its place is $(-y)f_4(b-1)$. Use this factor to weight the last brick such that the final cell contains y, the second to last cell contains $-y$, and the rest of the cells contain a sequence of x's and $-y$'s such that no two consecutive x's appear. These are the same objects as in the proof of Theorem 2.4 with the exception that the second to last cell in the last brick must contain $-y$.

Applying the weight-preserving sign-reversing involution as found in the proof of Theorem 2.4 to this set of objects, we are left with fixed points corresponding to permutations without 2-descents and the additional property that $\sigma_{n-1} < \sigma_n$. The powers of x and y on a fixed point register descents and rises. Therefore, we have

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \text{ without} \\ 2\text{-descents and } \sigma_{n-1} < \sigma_n}} x^{\operatorname{des}(\sigma)} y^{\operatorname{ris}(\sigma)} = \xi^{f_4}\left(\sum_{n=1}^{\infty} p_{n,\nu_4} t^n\right)$$
$$= \frac{\sum_{n=1}^{\infty}(-1)^{n-1}\nu_4(n)\xi^{f_4}(e_n)t^n}{\sum_{n=0}^{\infty}(-1)^n \xi^{f_4}(e_n)t^n}$$
$$= \frac{y\sum_{n=1}^{\infty} t^n f_4(n-1)/n!}{\sum_{n=0}^{\infty} t^n f_4(n)/n!}.$$

The expression in the denominator was simplified in the proof of Theorem 2.4. Using the result, the above string of equalities is equal to

$$\frac{y\mathbf{e}^{ty/2}}{\cos\left(\frac{t\sqrt{4xy-y^2}}{2}\right) - \frac{y}{\sqrt{4xy-y^2}}\sin\left(\frac{t\sqrt{4xy-y^2}}{2}\right)} \times \int \mathbf{e}^{-ty/2}\left(\cos\left(\frac{t\sqrt{4xy-y^2}}{2}\right) - \frac{y}{\sqrt{4xy-y^2}}\sin\left(\frac{t\sqrt{4xy-y^2}}{2}\right)\right) dt.$$

Integrating, this is equal to

$$\frac{y\mathbf{e}^{ty/2}}{\cos\left(\frac{t\sqrt{4xy-y^2}}{2}\right) - \frac{y}{\sqrt{4xy-y^2}}\sin\left(\frac{t\sqrt{4xy-y^2}}{2}\right)} \times \frac{1}{\sqrt{4xy-y^2}} 2\mathbf{e}^{-ty/2}\sin\left(\frac{t\sqrt{4xy-y^2}}{2}\right),$$

which may be seen to equal to the generating function in the statement of the theorem. □

The generating function for those permutations without 2-descents and $\sigma_{n-1} > \sigma_n$ can be found by taking the difference between the generating functions in Theorem 2.4 and Theorem 2.12.

A valley of a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is an index i between 2 and $n-1$ such that $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$. Let $\operatorname{val}(\sigma)$ be the number of valleys in σ. We will now show that we can find a generating function for the number of valleys in a permutation through Theorem 2.12 and a simple bijection. Later we will provide a different method of finding this generating function which more directly uses our methods.

THEOREM 2.13. *The number of permutations of n with k valleys is equal to 2^{n-2k-1} times the number of permutations $\sigma \in S_n$ without 2-descents, $\sigma_{n-1} < \sigma_n$, and* $\operatorname{des}(\sigma) = k$.

PROOF. Let $\sigma \in S_n$ be a permutation with $\operatorname{val}(\sigma) = k$. Let σ_i be an integer in the permutation such that such that $\sigma_i > \sigma_{i+1}$ and $i - 1 = 0$ or $\sigma_{i-1} < \sigma_{i+1}$. Take j be the largest index such that $\sigma_i > \sigma_{i+1} > \cdots > \sigma_{i+j}$. If $i + j \neq n$, underline $\sigma_{i+1}, \ldots, \sigma_{i+(j-1)}$ and rearrange σ such that each of the now underlined integers appear in increasing order immediately before σ_i. If $i + j = n$, apply the same action to $\sigma_{i+1}, \ldots, \sigma_{i+j}$. Any integers already in increasing order immediately before σ_i should be used together with the underlined integers to form an increasing sequence. Perform this action to at every possible σ_i with the needed property to produce a permutation $\overline{\sigma}$. For example, suppose

$$\sigma = 10\ 9\ 1\ 8\ 11\ 2\ 7\ 5\ 6\ 13\ 18\ 17\ 12\ 4\ 14\ 16\ 13\ 3.$$

The σ_i such that $\sigma_i > \sigma_{i+1}$ and $i-1=0$ or $\sigma_{i-1} < \sigma_{i+1}$ are found above at $10, 11, 7, 18$, and 16. We have

$$\overline{\sigma} = \underline{9}\ 10\ 1\ 8\ 11\ 2\ 7\ 5\ 6\ \underline{12}\ 13\ \underline{17}\ 18\ \underline{3}\ 4\ \underline{13}\ 14\ 16.$$

For each descent in $\overline{\sigma}$ there are two integers which cannot be underlined—the integers i and j in σ which begin and end a decreasing sequence. In addition, the last integer in $\overline{\sigma}$ cannot be underlined. Therefore, underlines in $\overline{\sigma}$ can possibly appear in only $n-2k-1$ places.

It is not difficult to see that at for each valley in σ, one and only one descent occurs in $\overline{\sigma}$. We have forced $\overline{\sigma}$ to have $\mathrm{des}(\overline{\sigma}) = k$, no 2-descents, and $\sigma_{n-1} < \sigma_n$. Furthermore, it is not difficult to see that this process is $1-1$. This completes the proof as we have displayed a bijection between the set of permutations of n with k valleys and the permutations $\sigma \in S_n$ without 2-descents, $\sigma_{n-1} < \sigma_n$, and $\mathrm{des}(\sigma) = k$ where possibly $n-2k-1$ integers are either underlined or not. □

COROLLARY 2.14.

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{val}(\sigma)} = \frac{1}{\sqrt{x-1}\cot(t\sqrt{x-1}) - 1}.$$

PROOF. We have

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{val}(\sigma)} = \sum_{n=1}^{\infty} \frac{t^n}{n!} 2^{n-1} \sum_{\substack{\sigma \in S_n \text{ without} \\ \text{2-descents and } \sigma_{n-1} < \sigma_n}} \left(\frac{x}{4}\right)^{\mathrm{des}(\sigma)}$$
$$= \frac{1}{2} \frac{2}{\sqrt{4\frac{x}{4}-1}\cot\left(\frac{2t}{2}\sqrt{4\frac{x}{4}-1}\right) - 1}$$

where the last line follows from Theorem 2.12. □

Corollary 2.14 was first proved by Entringer in 1969 via solving differential equations [**32**] (his notion of maxima is equivalent to our idea of valley). Carlitz published a few papers containing the result [**23, 24, 21**] and Gessel showed how this generating function fit into his framework [**42**]. In these publications the connection to the set of permutations $\sigma \in S_n$ without 2-descents and $\sigma_{n-1} < \sigma_n$ was not noted.

2.3. Consecutive patterns

Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\mathrm{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a τ-match at place i provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ.

To prevent confusion, we note that a permutation not having a τ-match different than a permutation being τ-avoiding. A permutation is called τ-avoiding if there are no indices $i_1 < \cdots < i_j$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. For example, if

$\tau = 2\ 1\ 4\ 3$, then the permutation $3\ 2\ 1\ 4\ 6\ 5$ does not have a τ-match but it does not avoid τ since $\mathrm{red}(2\ 1\ 6\ 5) = \tau$. For more details on τ-avoiding permutations, see [**12**].

Recent publications have analyzed the distribution of τ-matches in permutations and several nice theorems have been proved [**30, 49, 48**]. One specific result is as follows. Let $\tau\text{-nlap}(\sigma)$ be the maximum number of nonoverlapping τ-matches in σ where two τ-matches are said to overlap if they contain any of the same integers. It was published both in a paper and the doctoral dissertation by Sergey Kitaev [**49, 48**] that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{A(t)}{(1-x) + x(1-t)A(t)} \tag{2.10}$$

where $A(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$. In other words, if the exponential generating function for the number of permutations in S_n without any τ-matches is known, then so is the exponential generating function for the entire distribution of the statistic τ-nlap.

Suppose $\Upsilon \subseteq S_j$. We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ has an Υ-match at place i provided $\mathrm{red}(\sigma_i \cdots \sigma_{i+j-1}) \in \Upsilon$. Let $\Upsilon\text{-mch}(\sigma)$ and $\Upsilon\text{-nlap}(\sigma)$ be the number of Υ-matches and nonoverlapping Υ matches in σ, respectively. We will prove an analogue of (2.10) where every τ in (2.10) is replaced with an Υ.

It may be difficult to find the exponential generating function counting the number of permutations in S_n with $\tau\text{-mch}(\sigma) = 0$ and thus finding the generating function $A(t)$ in the statement of Kitaev's theorem may be difficult. Therefore, we will develop an alternate method of understanding the expression on the right hand side of (2.10).

After this is done, we will show how slight modifications to our new proof can yield results about inversion counts of permutations refined by the maximum number of nonoverlapping pattern matches. This will provide a q-analogue for (2.10). Using another variant on our proof, we will provide results about pattern matches for m-tuples of permutations.

With an eye toward proving Kitaev's Theorem, we now define a few auxiliary sets associated with a given permutation $\tau \in S_j$. For a permutation $\sigma \in S_n$, let

$$\mathrm{Mch}_\tau(\sigma) = \{i : \mathrm{red}(\sigma_{i+1} \cdots \sigma_{i+j}) = \tau\} \qquad \text{and}$$
$$I_\tau = \{1 \le i < j : \text{there exist } \sigma \in S_{j+i} \text{ such that } \mathrm{Mch}_\tau(\sigma) = \{0, i\}\}.$$

One (or one's computer) can find every element in the set I_τ for any $\tau \in S_j$ by finding the set $\mathrm{Mch}_\tau(\sigma)$ for all $\sigma \in S_{j+i}$ for $i = 1, \ldots, j-1$.

Let I_τ^* be the set of all words with letters in the set I_τ. We let ϵ denote the empty word. If $w = w_1 \cdots w_n \in I_\tau^*$ is word with n-letters, we define

$$\ell(w) = n, \qquad \sum w = \sum_{i=1}^{n} w_i, \qquad \text{and} \qquad ||w|| = j + \sum w.$$

In the special case where $w = \epsilon$, we let $\ell(w) = 0$ and $\sum w = 0$. Let

$$A_\tau = \{w \in I_\tau^* : \ell(w) \ge 2 \text{ and } \sum w < j\} \qquad \text{and}$$
$$B_{u,\tau} = \{w_1 \cdots w_n \in I_\tau^* : \sum w_2 \cdots w_n + \sum u < j \le \sum w_1 \cdots w_n + \sum u\}$$

for each word $u \in I_\tau^*$ with $\sum u < j$.

To help us understand these sets, let us look at the situation for four different permutations τ. These examples should provide insight into the kinds of situations which may arise.

First, consider $\alpha = 2\ 1\ 4\ 3$. We have that $\mathrm{Mch}_\alpha(2\ 1\ 4\ 3\ 6\ 5) = \{0, 2\}$ and $\mathrm{Mch}_\alpha(3\ 2\ 5\ 4\ 1\ 6\ 7) = \{0, 3\}$. Therefore $2, 3 \in I_\alpha$. A quick check can show that $1 \notin I_\alpha$; giving that $I_\alpha = \{2, 3\}$. The set I^*_α is the set of all words in the letters 2 and 3. Therefore $A_\alpha = \varnothing$ since every word w of length ≥ 2 in the letters 2 and 3 has $\sum w \geq 4$. The sets $B_{2,\alpha} = B_{3,\alpha}$ are both equal to $\{2, 3\}$ in this case.

Next, consider $\beta_j = j \cdots 2\ 1$. It may be shown that $I_{\beta_j} = \{1\}$, and therefore $A_{\beta_j} = \{1^k : 2 \leq k < j\}$ (here, 1^k denotes the word $1 \cdots 1$). If $1 \leq i < j$ then $B_{i,\beta_j} = \{1^{j-i}\}$.

When $\gamma = 2\ 1\ 4\ 3\ 6\ 5$, $I_\gamma = \{2, 5\}$, $A_\gamma = \{22\}$, $B_{2,\gamma} = B_{5,\gamma} = \{2, 5\}$ and $B_{22,\gamma} = \{22, 5\}$.

As our last example, let $\delta = 1\ 5\ 2\ 7\ 3\ 8\ 4\ 9\ 6$. Via a computer search, I_δ has been shown to equal $\{2, 4, 6, 8\}$. We are grateful to Jeff Liese; he found δ and I_δ. In this situation,

$$\begin{aligned}
A_\delta &= \{22, 24, 26, 42, 44, 62, 222, 224, 242, 422, 2222\},\\
B_{2,\delta} &= \{8, 26, 44, 62, 224, 242, 422, 2222\},\\
B_{4,\delta} &= B_{22,\delta} = \{6, 8, 24, 62, 42, 44, 222, 422\},\\
B_{6,\delta} &= B_{24,\delta} = B_{42,\delta} = B_{222,\delta} = \{4, 6, 8, 22\}, \qquad \text{and}\\
B_{8,\delta} &= B_{26,\delta} = B_{44,\delta} = B_{62,\delta} = B_{224,\delta} = B_{242,\delta} = B_{422,\delta} = \{2, 4, 6, 8\}.
\end{aligned}$$

Form a new alphabet

$$K_\tau = \{\overline{u} : u \in I_\tau\} \cup \{\overline{w} : w \in A_\tau\}.$$

We let $\Psi : K^*_\tau \to I^*_\tau$ be the function such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \dots w_n$. For example, if $\tau = \gamma = 2\,1\,4\,3\,5\,6$ as above, then $\Psi(\overline{5}\,\overline{22}\,\overline{2}\,\overline{2}\,\overline{5}) = 522225$.

Define $\overline{J}_\tau$ in the following manner.

(1) $\epsilon \in \overline{J}_\tau$.
(2) $\overline{v} \in \overline{J}_\tau$ for all $v \in I_\tau$.
(3) If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$, then $\overline{u}\ \overline{w_1} \cdots \overline{w_n} \in \overline{J}_\tau$ for all $u \in B_{w_1,\tau}$.
(4) The only words in $\overline{J}_\tau$ are the result of applying one of the above rules.

Take $J_\tau = \Psi(\overline{J}_\tau)$.

As an example, consider taking $\tau = \gamma = 2\ 1\ 4\ 3\ 5\ 6$. Then

$$\overline{J}_\gamma = \{\epsilon, \overline{2}, \overline{5}, \overline{22}\,\overline{2}, \overline{5}\,\overline{2}, \overline{2}\,\overline{5}, \overline{5}\,\overline{5}, \overline{2}\,\overline{22}\,\overline{2}, \overline{5}\,\overline{22}\,\overline{2}, \overline{2}\,\overline{5}\,\overline{2}, \overline{5}\,\overline{5}\,\overline{222}\,\overline{2}\,\overline{5}, \overline{5}\,\overline{2}\,\overline{5}, \overline{2}\,\overline{5}\,\overline{5}, \overline{5}\,\overline{5}\,\overline{5}, \ldots\}$$

The final set we would like to define is as follows. Let

$$\mathcal{P}^\tau_w = \{\sigma \in S_{||w||} : \mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}\}$$

and $\mathcal{P}^\tau_\epsilon = \{\tau\}$.

We will now make a key observation. Suppose that $\overline{u} = \overline{u}_k \cdots \overline{u}_1 \in \overline{J}_\tau$ and $\Psi(\overline{u}) = w_1 \cdots w_n$. Thus $w = w_1 \cdots w_n \in J_\tau$. Now suppose that $\sigma \in \mathcal{P}^\tau_w$. Then

$$\mathrm{Mch}_\tau(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}.$$

We can scan σ to discover that there are τ-matches at positions $1, 1 + w_1, 1 + w_1 + w_2, \ldots, 1 + w_1 + w_2 + \cdots + w_n$ so that we can recover $w_1, \ldots, w_n$ from σ. In addition, we claim that $\overline{u}$ can also be recovered. Let us now describe an algorithm which does this.

Step 1. Set $\overline{u}_1 = \overline{w}_n$. (This must be the case since the only words of length 1 in $\overline{J}_\tau$ are of the form $\overline{v}$ for $v \in I_\tau$.)
Step s+1. Suppose that we have recovered $\overline{u}_1, \ldots, \overline{u}_s$, $\Psi(\overline{u}_s \cdots \overline{u}_1) = w_b \cdots w_n$, and $\Psi(\overline{u}_s) = w_b \cdots w_c$ where $b \leq c$. It must be the case that $w_b + \cdots + w_c = r$ for some $1 \leq r \leq j-1$. If $w_{b-1} + r \geq j$, then set $\overline{u}_{s+1} = \overline{w}_{b-1}$. Otherwise, let a be the unique integer such that $1 \leq a < b-1$ such that $w_a + \cdots + w_c \geq j$, but $w_{a+1} + \cdots + w_c < j-1$ and set $\overline{u}_{s+1} = \overline{w_a \cdots w_{b-1}}$. (In this latter case, it follows from our definitions that $w_a \cdots w_{b-1}$ is the unique cofinal sequence of $w_1 \cdots w_{b-1}$ such that $w_a \cdots w_{b-1} \in B_{w_b \cdots w_c, \tau}$ so that it must be the case that $\overline{u}_{s+1} = \overline{w_a \cdots w_{b-1}}$.)

As an example, consider $w = 222252222225 \in J_\gamma$ where $\gamma = 2\ 1\ 4\ 3\ 6\ 5$. The algorithm would proceed as follows.

Step 1. $\overline{u}_1 = \overline{5}$.
Step 2. $\overline{u}_2 = \overline{2}$ since $2 + 5 \geq 6$.
Step 3. $\overline{u}_3 = \overline{22}$ since $(2+2) + 2 \geq 6$.
Step 4. $\overline{u}_4 = \overline{2}$ since $2 + (2+2) \geq 6$.
Step 5. $\overline{u}_5 = \overline{22}$ since $(2+2) + 2 \geq 6$.
Step 6. $\overline{u}_6 = \overline{5}$ since $5 + (2+2) \geq 6$.
Step 7. $\overline{u}_7 = \overline{2}$ since $2 + 5 \geq 6$.
Step 8. $\overline{u}_8 = \overline{22}$ since $(2+2) + 2 \geq 6$.
Step 9. $\overline{u}_9 = \overline{2}$ since $2 + (2+2) \geq 6$.

Thus $w = \Psi(\overline{2}\ \overline{22}\ \overline{2}\ \overline{5}\ \overline{22}\ \overline{2}\ \overline{22}\ \overline{2}\ \overline{5})$. This given, for each word $w \in J_\tau$, let $\overline{\ell}(w)$ to be the length of the word $\overline{u} \in \overline{J}_\tau$ such that $\Psi(\overline{u}) = w$.

In order to find the exponential generating function for the number of permutations in S_n refined by the maximum number of nonoverlapping τ-matches, we will introduce a homomorphism on the ring of symmetric functions by defining it on e_n for $n \geq 0$. Let $f_{10}(n) = (-1)^n$ if $n = 0, 1$ and

$$f_{10}(n) = (1-x) \sum_{\omega \in J_\tau, ||\omega|| = n} (-1)^{\overline{\ell}(\omega)} |\mathcal{P}_\omega^\tau|$$

otherwise. Let $\xi^{f_{10}}$ be the ring homomorphism on Λ with the property that

$$\xi^{f_{10}}(e_n) = \frac{(-1)^n}{n!} f_{10}(n).$$

The application of the homomorphism $\xi^{f_{10}}$ on $n!h_n$ is the key to understanding our proof of Theorem 2.15.

THEOREM 2.15 ([**49**]). *For $\tau \in S_j$,*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-}nlap(\sigma)} = \frac{A(t)}{(1-x) + x(1-t)A(t)},$$

where $A(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-}mch(\sigma) = 0\}|$.

PROOF. We have

$$\begin{aligned} n!\xi^{f_{10}}(h_n) &= n!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\xi^{f_{10}}(e_\lambda) \\ &= n!\sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\prod_{i=1}^{\ell(\lambda)}\frac{(-1)^{\lambda_i}}{\lambda_i!}f_{10}(\lambda_i) \\ &= \sum_{\lambda\vdash n}\binom{n}{\lambda}(-1)^{\ell(\lambda)}B_{\lambda,n}f_{10}(\lambda_1)\cdots f_{10}(\lambda_\ell). \end{aligned} \tag{2.11}$$

From (2.11), we will build a set $\mathcal{T}_\tau$ of decorated weighted brick tabloids.

The summand in (2.11) selects a partition λ of n. Use the $B_{\lambda,n}$ term to choose a brick tabloid T of shape (n) filled with bricks $b_1,\ldots,b_\ell$, reading from left to right, that induce the partition λ. With the multinomial coefficient, select pairwise disjoint subsets of $\{1,\ldots,n\}$ of size $|b_i|$ to assign to the bricks b_i for $i=1,\ldots,\ell$. At this point, we are left with factors of the form $(-1)^{\ell(\lambda)}$ and $f_{10}(\lambda_1)\cdots f_{10}(\lambda_\ell)$ to aid in the construction of elements in the set $\mathcal{T}_\tau$.

We assign a power of -1 coming from the factor of the form $(-1)^{\ell(\lambda)}$ in dealing with a brick of length 1. The factor of $f_{10}(1)$ which must appear in $f_{10}(\lambda_1)\cdots f_{10}(\lambda_\ell)$ provides an additional power of -1. These two powers of -1 cancel each other out. Thus, when using the terms in (2.11) to fill the cell in a brick of length 1, we simply place the assigned integer between 1 and n and nothing else.

For each brick b_i with $|b_i|>1$, we have a power of -1 coming from the term $(-1)^{\ell(\lambda)}$ and one $f_{10}(|b_i|)$ term. We use the $f_{10}(|b_i|)$ term to do the following things. First, pick $w\in J_\tau$ such that $||w||=|b_i|$ and select $\sigma\in\mathcal{P}^\tau_w$. Reorder the elements assigned to b_i so that the numbers are equal to σ when written as a permutation of $1,\ldots,|b_i|$. If $w=i_1i_2\cdots i_k$, then let $\overline{u}=\overline{u}_1\cdots\overline{u}_t$ be the word in $\overline{J}_\tau$ such that $\Psi(\overline{u})=w$ and let $u_i=\Psi(\overline{u}_i)$ and $j_i=\sum u_i$ for $i=1,\ldots,t$. Place a -1 on top of the cells $j_1,j_1+j_2,\cdots,j_1+j_2+\cdots j_t$ in b_i. This accounts for the $(-1)^{\overline{\ell}(w)}$ term. Finally, the product of the -1 coming from the $(-1)^{\ell(\lambda)}$ and the term $(1-x)$ coming from $f_{10}(|b_i|)$, leaves us with an $x-1$ term. Thus we make the choice of either placing an x or a -1 on the last cell in the brick. Our definitions ensure that we can recover w from σ.

To re-cap, our construction gives a that elements $T\in\mathcal{T}_\tau$ are brick tabloids filled such that

- each integer between 1 and n appears once in T,
- a brick of length 1 contains one integer,
- a brick b of length $m\geq 2$ contains an ordered sequence of integers which reduces to σ for some $\sigma\in\mathcal{P}^\tau_w$ such that $w=i_1i_2\cdots i_k\in J_\tau$ and $||w||=m$. We can then find $\overline{u}=\overline{u}_1\cdots\overline{u}_t\in\overline{J}_\tau$ such that $\Psi(\overline{u})=w$, and set $u_i=\Psi(\overline{u}_i)$ and $j_i=\sum u_i$ for $i=1,\ldots,t$. Then there are -1 on top of the cells $j_1,j_1+j_2,\ldots,j_1+j_2+\cdots j_t$ in b and a choice of either x or -1 for the terminal cell of b.
- if there is no $w\in J_\tau$ such that $||w||=m$, then there are no bricks of length m.

For instance, suppose that $\tau=1\ 3\ 2$. Then it is easy to see that $I_\tau=\{2\}$ and $J_\tau=I^*_\tau=\{2\}^*$. A brick tabloid $T\in\mathcal{T}_\tau$ may be found below.

		-1			x					-1	
7	3	11	4	10	8	6	1	5	12	9	2

Define the weight of $T \in \mathcal{T}_\tau$, $w(T)$, to be the product of all of the powers of x and -1 in the tabloid. From our construction, $n!\xi^{f_{10}}(h_n) = \sum_{T \in \mathcal{T}_\tau} w(T)$.

Next we define a sign-reversing involution $\mathfrak{I}_\tau$ which preserves the powers of x on a brick tabloid as follows. Scan the bricks of $T \in \mathcal{T}_\tau$ from left to right looking for the first of the following situations:

> *Case 1.* j consecutive bricks of length one such that the integers in these j bricks form a τ-match,
>
> *Case 2.* a brick b of length j with a weight of -1,
>
> *Case 3.* i bricks of length 1 followed by a brick b of length $m \geq 2$ such that
>
> > (i) b contains a sequence of integers which reduces to some $\sigma \in \mathcal{P}_w^\tau$ where $w = w_1 w_2 \cdots w_k \in J_\tau$ and $||w|| = m$,
> >
> > (ii) there is a word $\overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}_1 \cdots \overline{u}_r) = w_1 w_2 \cdots w_k$ and $\Psi(\overline{u}_1) = u_1 \in A_\tau$.
> >
> > (iii) if one concatenates the integers in the i bricks before b with the ordered sequence in b, this sequence reduces to some $\alpha \in P_v$ where $v = v_1 \cdots v_i w_1 \cdots w_k \in J_\tau$ and there is a word $\overline{u}\, \overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}\, \overline{u}_1 \cdots \overline{u}_r) = v_1 \cdots v_i w_1 \cdots w_k$ and $\Psi(\overline{u}) = v_1 \cdots v_i \in B_{\tau, u_1}$.
>
> *Case 4.* a brick b' of length $m > j$.

We then take the following actions. In Case 1, define $\mathfrak{I}_\tau(T) = T'$ be the result of combining the j bricks of size 1 in T to form one brick b of length j in T' that contains a τ-match. We also place a weight of -1 in the terminal cell of b.

In Case 2, we let $\mathfrak{I}_\tau(T) = T'$ be the result of splitting the brick b of length j in T that contains a τ-match into j bricks of size one in T'.

In Case 3, we let $\mathfrak{I}_\tau(T) = T'$ be the result of combining the first i cells to the left of the brick b plus the brick b into a single brick b' in T'. Thus there is a word $v = v_1 \cdots v_i w_1 \cdots w_k \in J_\tau$ and an $\alpha \in P_v^\tau$ such that the integers in b' reduce to give α. In addition, we place a -1 on the $(\sum v_1 \cdots v_i)$-th cell of b' can keep the labels on the cells that came from b the same.

In Case 4, suppose that $w = w_1 w_2 \cdots w_k \in J_\tau$ is the word of length ≥ 1 such that there is a $\gamma \in P_w^\tau$ for which the integers in b' are a γ-match. Suppose that $\overline{u} = \overline{u}_1 \cdots \overline{u_r}$ is the word in $\overline{J}_\tau$ such that $\Psi(\overline{u}) = w$ and $\Psi(\overline{u}_1) = u_1$ and $\sum u_1 = i$. Then we let $\mathfrak{I}_\tau(T) = T'$ be the result of splitting off the first i cells of b' into i bricks of size 1 and having the remaining cells form a single brick b. We keep that labels on the cells that are in b the same as they were in b'. It follows that $w' = \Psi(\overline{u}_2 \cdots \overline{u}_r)$ is in J_τ and there is $\sigma \in P_{w'}^\tau$ such that the integers in b reduce to σ.

For example, it is easy to see that the element in $\mathcal{T}_\tau$ for $\tau = 1\ 3\ 2$ previously displayed in this proof is in case 4 where b is the second brick. The sequence $3\ 11\ 4\ 10\ 8 \in P_2$ so that $w = 2 = \Psi(\overline{2})$. Thus $\overline{u}_1 = \overline{2}$, $\Psi(\overline{u}_1) = u_1 = 2$, and $i = \sum u_1 = 2$. Hence we break off the first two cells of b to form bricks of size 1 and we get the following configuration.

					x					-1	
7	3	11	4	10	8	6	1	5	12	9	2

First we will show that $\mathfrak{I}_\tau$ is a well defined involution. Suppose that we are in case 1 and that $\mathfrak{I}_\tau(T) = T'$ is the result of combining the j bricks of length 1 in T to form one brick b of length j in T' that contains a τ-match and put a -1 at the end of b. We claim that T' must be in Case 2 using the brick b.

After combining these j bricks of size 1 to form b, we can not be in case 1 since otherwise there are j bricks of size 1 to the left of b which form a τ-match. Similarly, we cannot be in cases 2, 3, or 4 using bricks that are strictly to left of b because such bricks existed in T and, since we always scan the bricks of T from left to right to find which of the four cases apply, we would have used such bricks to define $\mathfrak{I}_\tau(T)$.

Thus the definition of $\mathfrak{I}_\tau(T')$ must involve the brick b some how so that there are only two possibilities. Either we are in Case 2 and we split the brick b into j cells of size 1 so that $\mathfrak{I}_\tau(T') = T$ as desired or we are in Case 3 and we combine the i bricks of size 1 immediately preceding b together with b to form a new brick b'. We claim that this latter situation cannot happen. That is, we know that the ordered sequence in b forms a τ-match. It cannot be the case that there is an $i \in I_\tau$ such that the i bricks of length 1 immediately preceding b can be combined into form a single brick b' according to case 3. For this would mean that the i bricks of size 1 immediately preceding b plus the next $j-i$ bricks of size 1 formed a τ-match in T so that we would have used this set of bricks of size 1 to form a j-brick with τ-match in T rather than forming b.

Next, suppose that T is in Case 2 and we break the brick b of size j in T with a -1 at the end into j bricks of size 1 to form $T' = \mathfrak{I}_\tau(T)$. Then once again we cannot be any case for T' where we are using bricks which are all strictly to the left of b since we could have used those bricks to define $\mathfrak{I}_\tau(T)$ in the first place. Thus, whatever action that we take to form $\mathfrak{I}_\tau(T')$, it must involve the cells of b. In this case, there is only two such actions, namely, we can recombine the bricks of size 1 in T' that correspond to b into a brick of size b in which case $\mathfrak{I}_\tau(T') = T$ as desired or there is i bricks of size 1 preceding the cells of corresponding to b and we combine those i bricks of size 1 plus the first $j-i$ cells of size 1 corresponding to the brick b in T' to form a new brick b' which contains a τ-match.

We claim that the latter case cannot happen. Otherwise, we can find the smallest s such that first s bricks of size preceding the cells corresponding to b in T' plus the first $j-s$ bricks of size 1 corresponding to the cells of b form a τ-match. It would follow that the ordered sequence δ consisting of the elements in those s bricks of size 1 preceding b followed by the elements in the j bricks of size 1 corresponding the cells of b would form a sequence with exactly two τ matches, one starting at position 1 and one starting at position $s+1$. It would follow that $s \in I_\tau$ that δ is σ-match for some $\sigma \in P_s^\tau$. But in that case, it is easy to see that we could have used those s bricks of size 1 preceding b and b to see that Case 3 would apply to T. Since this is an action that we could have taken in T using bricks that are to the left of b, we would have taken that action rather split b into j bricks of size 1. Thus, given our assumptions for $\mathfrak{I}_\tau(T)$, there can be no such i so that it must be the case that $\mathfrak{I}_\tau(T') = T$.

Next suppose that we are in case 3. First we must show that i is uniquely determined by b. Let $w = w_1 \cdots w_k$ be the word in J_τ such that there is a $\sigma \in \mathcal{P}_w^\tau$ and the ordered sequence in b is a σ-match. Let $\overline{u}_1 \cdots \overline{u}_r$ be the word in $\overline{J}_\tau$ such that $\Psi(\overline{u}_1 \cdots \overline{u}_r) = w$ and suppose that $\Psi(\overline{u}_1) = w_1 \cdots w_a \in A_\tau$. Now suppose that $i_1 < i_2$ are integers such that

(1) if one concatenates the integers in the i_1 bricks before b with the ordered sequence in b, this sequence reduces to α for some $\alpha \in P^\tau_v$ where $v = v_1 \cdots v_i w_1 \cdots w_k \in J_\tau$ and there is a word $\overline{u}\ \overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}\ \overline{u}_1 \cdots \overline{u}_r) = v_1 \cdots v_i w_1 \cdots w_k$ and $\Psi(\overline{u}) = v_1 \cdots v_i \in B_{\tau,u_1}$, and

(2) if one concatenates the integers in the i_2 bricks before b with the ordered sequence in b, this sequence reduces to γ for some $\gamma \in \mathcal{P}^\tau_{v'}$ where $v' = v'_1 \cdots v'_s w_1 \cdots w_k \in J_\tau$ and there is a word $\overline{u'}\ \overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}\ \overline{u}_1 \cdots \overline{u}_r) = v_1 \cdots v_i w_1 \cdots w_k$ and $\Psi(\overline{u'}) = v'_1 \cdots v'_s \in B_{\tau,u_1}$.

It is not difficult to see that there must be an α-match in γ starting at position $i_2 - i_1 + 1$. This means that

$$v' = v'_1 \cdots v'_t v = v'_1 \cdots v'_t v_1 \cdots v_i w_1 \cdots w_k$$

for some $t \leq s$. But one can see from our algorithm to compute Ψ^{-1}, there is the unique cofinal sequence of $v'_1 \cdots v'_t v_1 \cdots v_i$ which is in $B_{w_1 \cdots w_a,\tau}$ so that it cannot be the case that both $v'_1 \cdots v'_t v_1 \cdots v_i$ and $v_1 \cdots v_i$ are in $B_{w_1 \cdots w_a,\tau}$. Thus there cannot be such i_1 and i_2 and, hence, i is uniquely determined by b.

Therefore, we can assume that

(i) b contains an ordered sequence of integers which reduces to σ-match $\sigma \in \mathcal{P}^\tau_w$ where $w = w_1 w_2 \cdots w_k \in J_\tau$ and $||w|| = m$,
(ii) there is a word $\overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}_1 \cdots \overline{u}_r) = w_1 w_2 \cdots w_k$ and $\Psi(\overline{u}_1) = u_1 = w_1 \cdots w_a \in A_\tau$.
(iii) if one concatenates the integers in the i bricks before b with the ordered sequence in b, this sequence forms a α-match for some word $\alpha \in P^\tau_v$ where $v = v_1 \cdots v_i w_1 \cdots w_k \in J_\tau$ and there is a word $\overline{u}\ \overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}\ \overline{u}_1 \cdots \overline{u}_r) = v_1 \cdots v_i w_1 \cdots w_k$ and $\Psi(\overline{u}) = v_1 \cdots v_i \in B_{\tau,w_1 \cdots w_a}$.

Then $T' = \mathfrak{I}(T)$ is formed by combining the i bricks of size one preceding b to form a new brick b' and then we put a -1 on the i-th cell of b' and keeping the labels on the remaining cells in b' the same as they were in b.

After applying case 3, we cannot be in Case 1 since otherwise there are j consecutive bricks of size 1 to left of b' which form a τ-match. But then these j consecutive bricks of size 1 were in T which violates the fact that case 3 applied T using the brick b. By a similar reasoning, we can not be in case 2 after applying case 3. We claim that we cannot be in case 3 either. That is, there cannot be a brick to the left of b' that we could use for case 3 since we would have used that brick in T in the first place. Moreover, we cannot be in case 3 where we combine i' bricks of size 1 immediately to left of b' with b' to form new brick b''. For in such a situation, the ordered sequence in b'' would be a β-match where $\beta \in P^\tau_z$ where $z = u_1 \cdots u_q v_1 \cdots v_i w_1 \cdots w_k$ is a word in J_τ and there is a word $\overline{u'}\ \overline{u}\ \overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that

$$\Psi(\overline{u'}\ \overline{u}\ \overline{u}_1 \cdots \overline{u}_r) = u_1 \cdots u_q v_1 \cdots v_i w_1 \cdots w_k$$

and $\Psi(\overline{u'}) = u_1 \cdots u_q \in B_{\tau,v_1 \cdots v_i}$. But by our definition of $\overline{J}_\tau$, it must be the case that

$$\sum \Psi(\overline{u'}\ \overline{u}) = \sum u_1 \cdots u_q v_1 \cdots v_i \geq j.$$

Thus before we created b', there were j consecutive bricks of size 1 starting $i'+i$ cells to the left of b in T whose integers formed a τ-match. But in such a situation, these j-consecutive bricks could have been used to apply case 1 to T so that we would not be in a situation where we applied case 3 to T using the brick b.

Finally, we cannot be in case 4 where we use some brick b'' to the left of b' because we could have used that brick in T. Thus we must be in case 4 where we are using the brick b'. Hence we would simply split off the first i cells of b' to form bricks of size 1 which would just recreate T. Thus in case 3, we will have $\mathfrak{I}_\tau(T') = T$.

Lastly, suppose we are in Case 4 and there is a word $w = w_1w_2\cdots w_k \in J_\tau$ of length ≥ 1 and $\gamma \in P^\tau_w$ such that the integers in b' form a γ-match. Let $\overline{u}_1\cdots\overline{u_r}$ be the word of $\overline{J}_\tau$ such that $\Psi(\overline{u}_1\cdots\overline{u_r}) = w_1w_2\cdots w_k \in J_\tau$. Suppose that $\Psi(\overline{u}_1) = w_1\cdots w_t$ where $t \leq k$ and that $\sum w_1\cdots w_t = i$.

Then $\mathfrak{I}_\tau(T) = T'$ is the result of splitting off the first i cells of b' into i bricks of length 1 and letting the remaining cells be a single brick b. We keep that labels on the cells that are in b the same as they were in b'. It follows that $w' = w_{a+1}\cdots w_k$ is in J_τ and there is $\sigma \in P^\tau_{w'}$ such that the integers in b form a σ-match.

We claim that T' cannot be in case 1. That is, there cannot be j bricks of size 1 that form a τ-match in T' that are strictly to the left of cells corresponding to b' since those bricks would have been in T and we would not have been in case 4 using b' for $\mathfrak{I}(T)$. Thus if there are such j bricks of size 1 in T', the cells corresponding to those bricks must intersect the i bricks of size 1 that we broke off from b' but not intersect any of the cells in b.

Say these j bricks of size 1 include the first c of the cells corresponding to b' so that $c \leq i$. Let δ be the order sequence of integers that results by taking the elements those $j-c$ bricks of size 1 immediately preceding b in T followed by the ordered sequence in b. Consider $\mathrm{Mch}_\tau(\mathrm{red}(\delta))$. Clearly,

$$0, j-c, j-c+w_1, j-c+(w_1+w_2), \ldots, j-c+(w_1+\cdots w_k) \in \mathrm{Mch}_\tau(\mathrm{red}(\delta)).$$

The only other elements of $\mathrm{Mch}_\tau(\mathrm{red}(\delta))$ must be of the form $0 < a_1 < \cdots < a_d < j-c$ so let $e_1 = a_1, e_2 = a_2 - a_1, \ldots, e_d = a_d - a_{d-1}, e_{d+1} = j-c-a_d$. Note if $d=0$, then we set $e_1 = j-c$. Note that $e_1 \in I_\tau$ since if one considers the reduced sequence formed by the first $j+e_1$ elements of δ, it will have τ-matches starting at positions 1 and e_1+1. Since $e_1 < j$, it follows that $e_1 \in I_\tau$. Similarly, if e_2 exists, then $e_2 \in I_\tau$ because if one considers the reduced sequence formed by the first $j+e_2$ elements of δ starting at position e_1+1, it will have τ-matches starting at positions 1 and e_2+1. Since $e_2 < j$, it follows that $e_2 \in I_\tau$. We can continue this line of argument to show that $e_i \in I_\tau$ for each $i = 1, \ldots, d+1$. By construction, $w_1\cdots w_a \in A_\tau$. So let s be such that

$$\sum e_{s+1}\cdots e_{d+1}\ w_1\cdots w_a < j \leq \sum e_s \cdots e_{d+1}\ w_1\cdots w_a.$$

It follows that $e_s\cdots e_{d+1} \in B_{\tau,w_1\cdots w_a}$ so that $\overline{e_s\cdots e_{d+1}}\ \overline{u}_1\cdots\overline{u}_r \in \overline{J}_\tau$. But then if $i = \sum e_s\cdots e_{d+1}$, then we would be in the following situation in T.

(i) b contains an ordered sequence of integers which is a σ-match for some $\sigma \in \mathcal{P}^\tau_w$ where $w = w_1w_2\cdots w_k \in J_\tau$ and $||w|| = m$,
(ii) there is a word $\overline{u}_1\cdots\overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}_1\cdots\overline{u}_r) = w_1w_2\cdots w_k$ and $\Psi(\overline{u}_1) = u_1 = w_1\cdots w_a \in A_\tau$.

(iii) if one concatenates the integers in the i bricks before b with the ordered sequence in b, this sequence reduces to α for some $\alpha \in \mathcal{P}_v^\tau$ where $v = e_s \cdots e_{d+1} w_1 \cdots w_k \in J_\tau$ and there is a word $\overline{u}\,\overline{u}_1 \cdots \overline{u}_r \in \overline{J}_\tau$ such that $\Psi(\overline{u}\,\overline{u}_1 \cdots \overline{u}_r) = e_s \cdots e_{d+1} w_1 \cdots w_k$ and $\Psi(\overline{u}) = e_s \cdots e_{d+1} \in B_{\tau, w_1 \cdots w_a}$.

But this would mean that we could apply Case 3 to T using b and the first i bricks of size 1 preceding b which involves cells to the left of b. This violates our assumption that T is in Case 4 using the brick b. Thus we cannot be in Case 1.

We can not be in Case 2 or Case 4 for T because the brick that witness T' is in case 2 or Case 4 would have to be strictly to the left of the cells corresponding to b' and hence they could have been used for T. Thus we must be in case 3. In fact, we claim that we must be in case 3 using brick b since otherwise there is some brick b'' of length greater than 2 to the left of b' and some i'' bricks of size 1 immediately to the left of b'' that witness that case 3 applies to T'. But then b'' would have to occur strictly to the left of i bricks of size 1 immediately to the left of b which means that the also were part of T. However, that again would violate the fact that we were in case 4 for T using b'. Thus we must be in case 3 for T' using the brick b so that once again we can conclude that $\mathfrak{I}_\tau(T') = T$.

By construction, if $\mathfrak{I}_\tau(T) = T'$, then $w(T) = -w(T')$. Therefore we have just shown that

$$n!\xi^{f_{10}}(h_n) = \sum_{T \in \mathfrak{T}_\tau} w(T) = \sum_{T \in \mathfrak{T}_\tau, \mathfrak{I}_\tau(T) = T} w(T).$$

The fixed points of $\mathfrak{I}_\tau$ must be tabloids which have only bricks of length one or bricks of length j with a weight of x whose integers form a τ-match. Each such fixed point T may be associated with a permutation σ of n written in one line notation by reading the integers from left to right.

If $\tau\text{-mch}(\sigma) = 0$, then T must contain only bricks of size 1 and the weight of T equals $1 = x^{\tau\text{-nlap}(\sigma)}$. Otherwise, suppose that $\mathrm{Mch}_\tau(\sigma) = \{a_1 < a_2 < \cdots < a_r\}$. Now define $r_1 < r_2 < \cdots < r_s$ by induction as follows.

(i) $r_1 = a_1$.
(ii) If r_k is defined, then r_{k+1} equals a_u where u is the least v such that $r_k + j < a_v$ if such a v exists.
(iii) If r_k is defined and there is no v such that $r_k + j < a_v$, then we set $k = s$.

It is not difficult to see that $s = \tau\text{-nlap}(\sigma)$. We claim that in this case, the bricks of size j in T start precisely at positions $r_1 + 1, r_2 + 1, \ldots, r_s + 1$ so that once again $w(T) = x^{\tau\text{-nlap}(\sigma)}$.

First observe that it cannot be that T has no brick of size j since otherwise the entries in j bricks of size 1 which start at position $a_1 + 1$ would form a τ-match and witness that case 1 could be applied to T. Thus T would not be a fixed point of $\mathfrak{I}_\tau$. Consider the position of the left-most brick b_1 of size j in T. Clearly b_1 must start at position $a_k + 1$ for some $1 \leq k \leq r$ since the integers in b_1 form a τ-match. We claim that k must equal one. That is, if $k > 1$, then consider the position $a_{k-1} + 1$. If $a_k - a_{k-1} \geq j$, then there are j consecutive bricks of size 1 to the left of b_1 whose entries form a τ-match and hence T could not be a fixed point of $\mathfrak{I}_\tau$ since these j bricks of size would witness that case 1 could apply to T. Similarly, if $i = a_k - a_{k-1} < j$, then integers in the i-bricks of size 1 to left of b_1 plus the integers in b_1 form σ-match for some σ such that $\mathrm{Mch}_\tau(\sigma) = \{0, i\}$ so that

$i \in I_\tau$. Thus the i bricks of size 1 to the left of b_1 and b_1 could be used to show that case 3 applies to T so that T could not have been a fixed point.

Thus the left-most brick of size j in b_1 must start at position $a_1 + 1 = r_1 + 1$. If $s \geq 2$, then we claim that there must be a second brick of size j in T. Otherwise $a_1 + j < r_2 + 1$ and hence there a j consecutive bricks of size 1 in T, starting at position $r_2 + 1$, whose entries form a τ-match so that case 1 could apply to T. Thus there must be a second brick of size j in T and we let b_2 be the left-most brick of size j to the right of b_1. Clearly b_2 must start at position $a_k + 1$ for some $r_2 \leq k \leq r$ since the integers in b_2 form a τ-match.

We claim that k must equal r_2. That is, if $k > r_2$, then consider the position $a_{k-1} + 1$ which is a position to the left of b_2. If $a_k - a_{k-1} \geq j$, then there are j consecutive bricks of size 1 to the left of b_2 whose entries form a τ-match and hence T could not be a fixed point of $\mathfrak{I}_\tau$ since these j bricks of size would witness that Case 1 could apply to T. Similarly, if $i = a_k - a_{k-1} < j$, then integers in the i-bricks of size 1 to left of b_2 plus the integers in b_2 form a σ-match for some σ such that $\mathrm{Mch}_\tau(\sigma) = \{0, i\}$ so that $i \in I_\tau$. Thus the i bricks of size 1 to the left of b_2 and b_2 could be used to show that Case 3 applies to T so that T could not have been a fixed point. Thus b_2 must start at position $r_2 + 1$. Clearly, we can continue on with this type of argument to show that T must have bricks of size j starting at positions $r_1 + 1, \ldots, r_s + 1$.

On the other hand, it is easy to see that if $\mathrm{Mch}_\tau(\sigma) = \emptyset$, then the brick tabloid T consisting of all brick of size 1 whose entries, when read from left to right, yield σ is a fixed point of $\mathfrak{I}_\tau$. Similarly, if $\mathrm{Mch}_\tau(\sigma) = \{a_1 < a_2 < \cdots < a_r\}$ and $r_1 < r_2 < \cdots < r_s$ are defined for σ as described above, then the brick tabloid T whose entries, when read from right to left, yield σ and which has bricks of size j whose last cell is labeled with x starting at positions $r_1 + 1, \ldots, r_s + 1$ and whose remaining cells are covered by bricks of size 1 is a fixed point of $\mathfrak{I}_\tau$. Thus, it follows that

$$n!\xi^{f_{10}}(h_n) = \sum_{T \in \mathfrak{T}_\tau, \mathfrak{I}_\tau(T) = T} w(T) = \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)}$$

as desired.

One example of a fixed point when $\tau = 1\ 3\ 2$ follows.

			x							x	
7	3	11	4	10	8	6	1	5	12	9	2

At this point, we can apply $\xi^{f_{10}}$ to both sides of the identity in (1.4) to obtain

(2.12)
$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-}nlap(\sigma)} &= \frac{1}{1 + \sum_{n=1}^{\infty} \xi^{f_{10}}(e_n)} \\ &= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!}(1-x) \sum_{w \in J_\tau, ||w|| = n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}.\end{aligned}$$

Then setting $x = 0$ in the above equations,

$$\begin{aligned} A(t) &= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| \\ &= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}. \end{aligned}$$

Thus

$$\sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau| = \frac{1}{A(t)} - (1 - t). \tag{2.13}$$

Combining (2.12) and (2.13), we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} &= \frac{1}{1 + \sum_{n=1}^{\infty} \xi^{f_{10}}(e_n)} \\ &= \frac{1}{1 - t + (1-x)(\frac{1}{A(t)} - (1-t))} \\ &= \frac{A(t)}{(1-x) - x(1-t)A(t)}. \end{aligned}$$

□

Our proof of Kitaev's theorem gives the following corollaries.

Corollary 2.16. *For any permutation τ,*

$$\begin{aligned} A(t) &= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-}mch(\sigma) = 0\}| \\ &= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}. \end{aligned}$$

Corollary 2.17. *For any permutation τ,*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-}nlap(\sigma)} = \frac{1}{1 - t + (1-x) \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\tau, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\tau|}.$$

As a first example of the use of Theorem 2.15,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} = \frac{1}{x(1-t) + (1-x)\left(\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!} - \sum_{n=0}^{\infty} \frac{t^{jn+1}}{(jn+1)!}\right)}$$

provided $\tau = j \cdots 2\ 1$. Special cases of the above expression sometimes simplify nicely; for example, when $j = 2$:

$$\frac{\mathbf{e}^t}{(1-x) + x(1-t)\mathbf{e}^t},$$

and when $j = 3$:

$$\frac{\mathbf{e}^{t/2}}{x(1-t)\mathbf{e}^{t/2} + (1-x)\left(\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)}.$$

If the generating function for the number of permutations without any τ-matches is known, then Theorem 2.15 is able to refine all permutations of n by the maximum number of nonoverlapping τ-matches. However, Theorem 2.15 does not give any direct way to find the number of permutations without τ-matches and, in general, it is difficult to count the number of permutations σ in S_n with $\tau\text{-mch}(\sigma) = 0$. However, Corollary 2.16 provides an alternative approach to finding the number of permutations σ in S_n with $\tau\text{-mch}(\sigma) = 0$. That is, instead of counting the number of permutations in S_n with $\tau\text{-mch}(\sigma) = 0$, we may try to understand the sum on the right hand side of the statement of the corollary.

As an example of this phenomenon, suppose we wanted to find out more about the maximum number of nonoverlapping τ-matches when $\tau = 1\ 3\ 2$. The set I_τ contains only the integer 2. This greatly simplifies matters since we have that $J_\tau = I_\tau^* = \{2\}^*$. Suppose that $\sigma_1 \dots \sigma_{2n+3} \in P_{2^n}^\tau$. Since there is τ-match starting at position $2i+1$ for $i = 0, \dots, n$, it must be the case that $\sigma_{2i+1} < \sigma_{2i+2}, \sigma_{2i+3}$ for $i = 0, \dots, n$. It follows that $\sigma_1 = 1$ and $\sigma_3 = 2$. It is not difficult to see that σ_2 can be any element of $\{3, \dots, 2n+3\}$ and that $\text{red}(\sigma_3 \dots \sigma_{2n+3}) \in P_{2^{n-1}}^\tau$ if $n \geq 1$. It follows that $|P_{2^n}^\tau| = (2n+1)|P_{2^{n-1}}^\tau|$ if $n \geq 1$. Since $P_{2^0}^\tau = 1$, it follows by induction that $|P_{2^n}^\tau| = (2n+1)!! = (2n+1)(2n-1)\cdots 3 \cdot 1$ for $n \geq 0$. Thus, from Corollary, 2.16 we have that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \text{Mch}_{132}(\sigma) = \emptyset\}| &= \frac{1}{1 - t + \sum_{n=0}^{\infty} (-1)^n |P_{2^n}^{132}| \frac{t^{2n+3}}{(2n+3)!}} \\
&= \frac{1}{1 - t + \sum_{n=0}^{\infty} (-1)^n (\prod_{i=0}^{n} (2i+1)) \frac{t^{2n+3}}{(2n+3)!}} \\
&= \frac{1}{1 - \int exp(-t^2/2) dt}
\end{aligned}$$

(for the definition of $Mch_{132}(\sigma)$ see below). From Theorem 2.15, the generating function refining the permutations of n by the maximum number of nonoverlapping τ-matches is equal to

$$\left(1 - tx + (x-1) \int \mathbf{e}^{-t^2/2}\, dt\right)^{-1}$$

in the case $\tau = 1\ 3\ 2$.

Theorem 2.15 was stated in the way it appears in the literature, but our method of proof may be generalized. Specifically, we can keep track of the nonoverlapping matches in more than one permutation.

Let $\Upsilon \subseteq S_j$. We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n$ has an Υ-match starting at position $i+1$ if $\text{red}(\sigma_{i+1} \cdots \sigma_{i+j}) \in \Upsilon$. For example, if $\sigma = 5\ 3\ 4\ 1\ 6\ 2$ and $\Upsilon = \{3\ 1\ 2, 2\ 1\ 3\}$, then σ has an Υ-match starting at positions 1 since $\text{red}(5\ 3\ 4) = 3\ 1\ 2$ and an Υ-match starting at position 3 since $\text{red}(4\ 1\ 6) = 2\ 1\ 3$. For any given permutation σ, we define $\Upsilon\text{-}nlap(\sigma)$ to be the maximum number of non-overlapping Υ-matches that occur in σ. Let

$$\text{Mch}_\Upsilon(\sigma) = \{i : \text{red}(\sigma_{i+1} \cdots \sigma_{i+j}) = \tau\} \qquad \text{and}$$
$$I_\Upsilon = \{1 \leq i < j : \text{there exist } \sigma \in S_{j+i} \text{ such that } \text{Mch}_\tau(\sigma) = \{0, i\}\}.$$

For example, it is easy to see that if $\Upsilon = \{3\ 1\ 2, 2\ 1\ 3\}$, then $I_\Upsilon = \{2\}$.

Let I_Υ^* be the set of all words with letters in the set I_Υ. We let ϵ denote the empty word. If $w = w_1 \cdots w_n \in I_\tau^*$ is word with n-letters, we define

$$\ell(w) = n, \qquad \sum w = \sum_{i=1}^{n} w_i, \qquad \text{and} \qquad ||w|| = j + \sum w.$$

In the special case where $w = \epsilon$, we let $\ell(w) = 0$ and $\sum w = 0$. Let

$$A_\Upsilon = \{w \in I_\Upsilon^* : \ell(w) \geq 2 \text{ and } \sum w < j\} \qquad \text{and}$$
$$B_{u,\Upsilon} = \{w_1 \cdots w_n \in I_\Upsilon^* : \sum w_2 \cdots w_n + \sum u < j \leq \sum w_1 \cdots w_n + \sum u\}$$

for each word $u \in I_\Upsilon^*$ with $\sum u < j$.

We also set

$$\mathcal{P}_w^\Upsilon = \{\sigma \in S_{||w||} : \mathrm{Mch}_\Upsilon(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}\}.$$

For example, if $\Upsilon = \{3\ 1\ 2, 2\ 1\ 3\}$, then permutation $\sigma = 2\ 1\ 4\ 3\ 7\ 6\ 9\ 8\ 5\ 11\ 10$ has $\mathrm{Mch}_\Upsilon(\sigma) = \{0, 2, 4, 7\}$ so that σ is an element of $\mathcal{P}_{2\ 2\ 3}^\Upsilon$.

Form a new alphabet

$$K_\Upsilon = \{\overline{u} : u \in I_\Upsilon\} \cup \{\overline{w} : w \in A_\Upsilon\}.$$

We define the natural map $\Psi : K_\Upsilon^* \to I_\Upsilon^*$ such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \ldots w_n$. Define $\overline{J}_\Upsilon$ recursively as follows.

(1) $\epsilon \in \overline{J}_\Upsilon$.
(2) $\overline{v} \in \overline{J}_\Upsilon$ for all $v \in I_\Upsilon$.
(3) If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_\Upsilon$, then $\overline{u}\ \overline{w_1} \cdots \overline{w_n} \in \overline{J}_\Upsilon$ for all $u \in B_{w_1,\Upsilon}$.
(4) The only words in $\overline{J}_\Upsilon$ are the result of applying one of the above rules.

Let $J_\Upsilon = \Psi(\overline{J}_\Upsilon)$. Then once again, for each $w \in J_\Upsilon$, we can construct the unique word $\overline{u} = \overline{u_1} \cdots \overline{u}_t$ such that $\Psi(\overline{u}) = w$. We then let $\overline{\ell}(w) = \ell(\overline{u})$.

In order to find the exponential generating function for the number of permutations in S_n refined by the maximum number of nonoverlapping Υ-matches, we will introduce a homomorphism on the ring of symmetric functions by defining it on e_n for $n \geq 0$. Let $U(n) = (-1)^n$ if $n = 0, 1$ and

$$U(n) = (1 - x) \sum_{\omega \in J_\Upsilon, ||\omega|| = n} (-1)^{\overline{\ell}(\omega)} \left|\mathcal{P}_\omega^\Upsilon\right|$$

otherwise. Let ξ_Υ be the ring homomorphism on Λ with the property that

$$\xi_\Upsilon(e_n) = \frac{(-1)^n}{n!} U(n).$$

With these definitions, we can use the same proof as in Theorem 2.15 to show that

$$n!\xi_\Upsilon(h_n) = \sum_{\sigma \in S_n} x^{\Upsilon\text{-}nlap(\sigma)}.$$

Then we can use the same manipulations as we did in Theorem 2.15 to prove the following.

THEOREM 2.18. *For any set of permutations $\Upsilon \subseteq S_j$ where $j > 1$,*

$$\begin{aligned} A^\Upsilon(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \Upsilon\text{-mch}(\sigma) = 0\}| \\ &= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\Upsilon, ||w|| = n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\Upsilon|}, \end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\Upsilon\text{-nlap}(\sigma)} &= \frac{1}{1-t+(1-x)\sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\Upsilon, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\Upsilon|} \\ &= \frac{A^\Upsilon(t)}{(1-x)+x(1-t)A^\Upsilon(t)}.\end{aligned}$$

As a check of Theorem 2.18, consider the case where $\Upsilon = S_j$. In this case, it is easy to see that $I_\Upsilon = \{1\}$. One can also easily show that $J_\Upsilon = \{1^{kj} : k \geq 0\} \cup \{1^{kj+1} : k \geq 0\}$. However, in this case, it easy to see that if $w \in J_\Upsilon$ and $||w|| = m$, then $\mathcal{P}_w^\Upsilon = m!$ since for any $\sigma \in S_m$, $\mathrm{Mch}_\Upsilon(\sigma) = \{0, 1, \ldots, m-j-1\}$. It then follows that

$$\begin{aligned} A^\Upsilon(t) &= 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \Upsilon\text{-mch}(\sigma) = 0\}| \\ &= \frac{1}{1-t+\sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{w \in J_\Upsilon, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\Upsilon|} \\ &= \frac{1}{1-t+\sum_{n=0}^{\infty} t^{nj} - t^{nj+1}} \\ &= 1 + t + t^2 + \cdots + t^{j-1}. \end{aligned}$$

Counting the maximum number of nonoverlapping valleys in a permutation is one use of Theorem 2.18 which does not follow immediately from the statement of Theorem 2.15. A valley of a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is an index i between 2 and $n-1$ such that $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$. A permutation has a valley if and only if it contains either a 3 1 2-match or a 2 1 3-match. Thus we let $\Upsilon = \{3\,1\,2, 2\,1\,3\}$ so that $I_\Upsilon = \{2\}$ and $J_\Upsilon = \{2\}^*$. If $\sigma \in S_n$ is a permutation with no valleys, then σ must be of the form

$$\sigma = \sigma_1 < \sigma_2 < \cdots < \sigma_k = n > \sigma_{k+1} > \cdots > \sigma_n$$

and, hence, it follows that the number of permutations of n without any valleys is 2^{n-1} provided $n \geq 1$. Therefore, the function $A^\Upsilon(t)$ counting the number of permutations in S_n without any valleys is $(\mathbf{e}^{2t}+1)/2$. By Theorem 2.18, the exponential generating function for the maximum number of nonoverlapping valleys is

$$\frac{1}{1-tx+(x-1)\tanh(t)}. \tag{2.14}$$

More generally, we can count the maximum number of nonoverlapping occurrences of $\sigma_i > \cdots > \sigma_{i+j-1}$ together with $\sigma_{i+j-1} < \sigma_{i+j}$ in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$. That is, let Υ be the set of all permutations $\tau \in S_{j+1}$ such that $\tau_1 > \cdots > \tau_j$ and $\tau_j < \tau_{j+1}$. Then it is easy to prove that $I_\Upsilon = \{j\}$ and hence $J_\Upsilon = \{j\}^*$. A permutation without a τ-match for $\tau \in \Upsilon$ may be thought of as the concatenation of a permutation of $n-\ell$ without a $j \cdots 2\,1$-match together with a decreasing string of length ℓ. For example, when $j=3$, the permutation 9 1 7 8 4 5 2 11 12 10 6 3 is the concatenation of 9 1 7 8 4 5 2 11 12 and 10 6 3. By mandating that ℓ is a multiple of j, this concatenation is unique.

Since there is only one decreasing permutation of jn, the generating function for permutations which must be have length a multiple of j and decreasing is

$\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!}$. Thus, the exponential generating function counting the number of permutations S_n without a τ-match for $\tau \in \Upsilon$ is given by

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\ell=0}^{n} \binom{n}{\ell} \left((n-\ell)! \frac{1}{\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!} - \sum_{n=0}^{\infty} \frac{t^{jn+1}}{(jn+1)!}} \Bigg|_{t^{n-\ell}} \right) \left(\ell! \sum_{i=0}^{\infty} \frac{t^{ij}}{(ij)!} \Bigg|_{t^{\ell}} \right)$$

because the coefficient of $t^{n-\ell}$ in the first parentheses selects a permutation of $n-\ell$ without a $j\cdots 2\ 1$-match, the coefficient of t^{ℓ} in the second parentheses selects a decreasing permutation of length ℓ a multiple of j, and the binomial coefficient chooses which integers to pair with each of them. Simplifying the above expression, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \Upsilon\text{-mch}(\sigma) = 0\}| = \frac{\sum_{i=0}^{\infty} \frac{t^{ij}}{(ij)!}}{\sum_{n=0}^{\infty} \frac{t^{jn}}{(jn)!} - \sum_{n=0}^{\infty} \frac{t^{jn+1}}{(jn+1)!}}.$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\Upsilon\text{-nlap}(\sigma)} = \frac{\sum_{n=0}^{\infty} \frac{t^{nj}}{(nj)!}}{(1-xt)\left(\sum_{n=0}^{\infty} \frac{t^{nj}}{(nj)!}\right) + (x-1)\left(\sum_{n=0}^{\infty} \frac{t^{nj+1}}{(nj+1)!}\right)}.$$

The above expression simplifies to (2.14) when $j = 2$ as it should. Another simple special case of (2.14) is

$$\frac{\cos(t) + \cosh(t)}{(1-tx)(\cos(t) + \cosh(t)) + (x-1)(\sin(t) + \sinh(t))}$$

found when $j = 4$.

Let Υ be a subset of S_j for some $j > 2$. To prove our q-analogue of Theorem 2.18, we introduce a q-analogue of ξ_{Υ}. Let $f_q(n) = (-1)^n$ if $n = 0, 1$ and take

$$f_q(n) = (1-x) \sum_{\|\omega\|=n, \omega \in J_{\Upsilon}} (-1)^{\overline{\ell}(\omega)} \sum_{\sigma \in \mathcal{P}_{\omega}^{\Upsilon}} q^{inv(\sigma)}$$

otherwise. Define ξ_q as the ring homomorphism on Λ with the property that

$$\xi_q(e_n) = \frac{(-1)^n}{[n]_q!} f_q(n).$$

By taking $q = 1$, we have $\xi_q = \xi_{\Upsilon}$. The homomorphism ξ_q will be used to construct brick tabloids as in the proof of Theorem 2.18 with the addition that we will keep track of the q raised to the number of inversions of the permutation associated with the brick tabloid.

Theorem 2.19. *For any set of permutations $\Upsilon \subseteq S_j$ where $j > 1$,*

$$\begin{aligned} A_q^{\Upsilon}(t) &= \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n : Mch_{\Upsilon}(\sigma) = \emptyset} q^{inv(\sigma)} \\ &= \frac{1}{1 - t + \sum_{n=2}^{\infty} \frac{t^n}{[n]_q!} \sum_{w \in J_{\Upsilon}, ||w||=n} (-1)^{\overline{\ell}(w)} \sum_{\sigma \in \mathcal{P}_w^{\Upsilon}} q^{inv(\sigma)}} \end{aligned}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]!} \sum_{\sigma \in S_n} x^{\Upsilon - nlap(\sigma)} q^{inv(\sigma)} = \frac{A_q^{\Upsilon}(t)}{(1-x) + x(1-t)A_q^{\Upsilon}(t)}.$$

Proof. From the same steps which gave (2.11),

$$[n]_q!\xi_q(h_n) = \sum_{\lambda \vdash n} \begin{bmatrix} n \\ \lambda \end{bmatrix}_q (-1)^{\ell(\lambda)} B_{\lambda,n} f_q(\lambda_1)\cdots f_q(\lambda_\ell). \tag{2.15}$$

Start with a brick tabloid of shape (n) and type λ for some $\lambda \vdash n$ in order to account for the summand and the $B_{\lambda,n}$ term. Leonard Carlitz proved that if $\mathcal{R}(1^{\lambda_1},\ldots,\ell^{\lambda_\ell})$ is the set of rearrangements of λ_1 1's, λ_2 2's, etc., then $\begin{bmatrix} n \\ \lambda \end{bmatrix}_q = \sum_{r\in\mathcal{R}(1^{\lambda_1},\ldots,\ell^{\lambda_\ell})} q^{\mathrm{inv}(r)}$ [**19**]. We can associate a permutation σ_r with each $r \in \mathcal{R}(1^{\lambda_1},\ldots,\ell^{\lambda_\ell})$ by labeling the 1's from left to right with $1,\ldots,\lambda_1$, labeling the 2's from left to right with $\lambda_1,\ldots,\lambda_1+\lambda_2$, etc. It then easy to see that $inv(\sigma_r) = inv(r) = inv(\sigma_r^{-1})$ and that σ_r^{-1} consists of a list of the positions of the 1's in r, followed by a list of positions of the 2's in r in increasing order, etc. For example, if $r = 1\ 2\ 1\ 3\ 1\ 3\ 2\ 2\ 1$, then

$$\begin{aligned} r &= 1\ 2\ 1\ 3\ 1\ 3\ 2\ 2\ 1, \\ \sigma_r &= 1\ 5\ 2\ 8\ 3\ 9\ 6\ 7\ 4, \qquad \text{and} \\ \sigma_r^{-1} &= 1\ 3\ 5\ 9\ 2\ 7\ 8\ 4\ 6. \end{aligned}$$

Thus if we are given a sequence of bricks $b_1,\ldots,b_\ell$ of lengths $\lambda_1,\ldots,\lambda_\ell$ respectively, then the q-binomial coefficient $\begin{bmatrix} n \\ \lambda \end{bmatrix}_q$ allows us to associate a sequence of pairwise disjoint sets to each brick so that the union of these sets is $\{1,\ldots,n\}$ and, in addition, we can assign a weight to each integer m in a brick b_i which is q raised to the number integers in bricks $b_{i+1},\ldots,b_\ell$ which are smaller than m.

The $(-1)^{\ell(\lambda)} f_q(\lambda_1)\cdots f_q(\lambda_\ell)$ term in (2.15) acts in the same manner as in the proof of Theorem 2.15 by selecting a word ω such that $\|\omega\| = m$ for each brick of length m, placing in that brick a permutation in $\mathcal{P}_\omega^\Upsilon$ when written as a permutation of $1,\ldots,m$, and adding -1's and x's in the appropriate places. In addition, however, the extra q's in f_q give powers of q registering the number of inversions caused within each brick.

In short, we are able to form brick tabloids in the same way as in the proof of Theorem 2.15 with one notable exception: the powers of q in the definition of the homomorphism ξ_q allow us to place a power of q in every cell counting the number of inversions caused by the integer in that cell. Below we have included an example of one such brick tabloid decorated in this manner when $\tau = 1\ 3\ 2$.

		-1		-1			-1				x
7	2	1	4	3	8	6	10	9	5	12	11
q^6	q^1	q^0	q^4	q^0	q^2	q^1	q^2	q^1	q^0	q^1	q^0

The permutation in a brick tabloid remains unaltered after an application of the involution found in Theorem 2.15 since the involution only changed the configuration of bricks and labels but not the permutation of n in the brick tabloid. This means that the same involution may be applied to the weighted brick tabloids described above since the powers of q will not be affected by the involution. Fixed points correspond to elements in the symmetric group where the total power of x count the maximum number of nonoverlapping τ-matches and the total power of q

count the number of inversions. Thus, we have shown that

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\Upsilon\text{-}nlap(\sigma)} q^{inv(\sigma)} = \xi_q \left(\sum_{n=0}^{\infty} h_n t^n \right) = \xi_q \left(\sum_{n=0}^{\infty} e_n (-t)^n \right)^{-1}$$

and the statement of the theorem follows by evaluating the homomorphism ξ_q on e_n and cleaning up the result with similar steps as those used in the proof of Theorem 2.15. □

Although many generating functions found using Theorem 2.19 do not often have a closed form, there are a few simple examples. As for one, consider $\tau = 2\ 1$. There is only one permutation without any τ-matches and that permutation has no inversions. If we let $\mathbf{e}^t_{p,q} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!}$ be a q-analogue for the exponential function, then

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\tau\text{-nlap}(\sigma)} q^{\mathrm{inv}(\sigma)} = \frac{\mathbf{e}^t_{p,q}}{(1-x) + x(1-t)\mathbf{e}^t_{p,q}}.$$

We turn our attention to providing a result about pattern matches in multiple permutations. For $\Upsilon \subseteq S_j$, we define an a collection of permutations $\sigma^1, \ldots, \sigma^m \in S_n$ to have a Υ-common match at place i provided that there is a $\sigma \in \Upsilon$ such that $\mathrm{red}(\sigma^\ell_i \cdots \sigma^\ell_{i+j-1}) = \sigma$ for every $1 \leq \ell \leq m$. Let Υ-*commch*$(\sigma^1, \ldots, \sigma^m)$ be the total number of Υ-common matches and let Υ-*comnlap*$(\sigma^1, \ldots, \sigma^m)$ be the maximum number of nonoverlapping Υ-common matches in the collection of elements $\sigma^1, \ldots, \sigma^m \in S_n$. If Υ consists of a single permutation τ, then we use the notation τ-common match, τ-*commch*$(\sigma^1, \ldots, \sigma^m)$, and τ-*comnlap*$(\sigma^1, \ldots, \sigma^m)$ for Υ-common match, Υ-*commch*$(\sigma^1, \ldots, \sigma^m)$, and Υ-*comnlap*$(\sigma^1, \ldots, \sigma^m)$ respectively.

When $\tau = 2\ 1$, the number of τ-common matches has been referred to as the number of common descents and was first studied by Leonard Carlitz, Richard Scoville and Theresa Vaughan [**24, 26, 21**]. Later, this statistic was studied by Jean-Marc Fédou with Don Rawlings and Thomas Langley with Jeff Remmel; the latter authors used the same approach of manipulating the relationships between symmetric functions we are taking [**34, 35, 53**]. This paper, however, marks the first time τ-common matches have been studied for τ other than the permutation 2 1.

By analogy, let $\mathrm{Mch}_\Upsilon(\sigma^1, \ldots, \sigma^m)$ be the set

$$\{i : \mathrm{red}(\sigma^q_{i+1} \cdots \sigma^q_{i+j}) = \tau \text{ for all } 1 \leq q \leq m\}$$

and let I^m_Υ be the set

$$\{1 \leq i < j : \text{there exist } \sigma^1, \ldots, \sigma^m \in S_{j+i} \text{ such that } \mathrm{Mch}_\tau(\sigma^1, \ldots, \sigma^m) = \{0, i\}\}.$$

Let $(I^m_\Upsilon)^*$ be the set of all words with letters in the set I^m_Υ. Let

$$A^m_\Upsilon = \{w \in (I^m_\Upsilon)^* : \ell(w) \geq 2 \text{ and } \sum w < j\} \qquad \text{and}$$

$$B^m_{u,\Upsilon} = \{w_1 \cdots w_n \in (I^m_\Upsilon)^* : \sum w_2 \cdots w_n + \sum u < j \leq \sum w_1 \cdots w_n + \sum u\}$$

for each word $u \in (I^m_\Upsilon)^*$ with $\sum u < j$. We also set $\mathcal{P}^{\Upsilon,m}_w$ to be equal to the set

$$\{(\sigma^1, \ldots, \sigma^m) \in S^m_{||w||} : \mathrm{Mch}^m_\Upsilon(\sigma^1, \ldots, \sigma^m) = \{0, w_1, w_1 + w_2, \ldots, w_1 + \cdots + w_n\}\}.$$

Form a new alphabet

$$K^m_\Upsilon = \{\overline{u} : u \in I^m_\Upsilon\} \cup \{\overline{w} : w \in A^m_\Upsilon\}.$$

We define the map $\Psi : (K_\Upsilon^m)^* \to (I_\Upsilon^m)^*$ such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1}\cdots\overline{w_n})$ is equal to $w_1 \ldots w_n$. Define $\overline{J}_\Upsilon^m$ recursively as follows.

(1) $\epsilon \in \overline{J}_\Upsilon^m$.
(2) $\overline{v} \in \overline{J}_\Upsilon^m$ for all $v \in I_\Upsilon^m$.
(3) If $\overline{w_1}\cdots\overline{w_n} \in \overline{J}_\Upsilon^m$, then $\overline{u}\;\overline{w_1}\cdots\overline{w_n} \in \overline{J}_\Upsilon^m$ for all $u \in B_{w_1,\Upsilon}^m$.
(4) The only words in $\overline{J}_\Upsilon^m$ are the result of applying one of the above rules.

Let $J_\Upsilon^m = \Psi(\overline{J}_\Upsilon^m)$. Then once again, for each $w \in J_\Upsilon^m$, we can construct the unique word $\overline{u} = \overline{u_1}\cdots\overline{u}_t$ such that $\Psi(\overline{u}) = w$. We then let $\overline{\ell}(w) = \ell(\overline{u})$.

For example, if $\Upsilon = \{3\ 1\ 2, 2\ 1\ 3\}$ and $m = 2$, then the pair

$$\sigma^1 = 2\ 1\ 4\ 3\ 7\ 6\ 9\ 8\ 5\ (11)\ (10) \qquad \text{and} \qquad \sigma^2 = 2\ 1\ 5\ 3\ 9\ 6\ 7\ 8\ 4\ (10)\ (11)$$

has $\mathrm{Mch}_\Upsilon^2(\sigma_1, \sigma_2) = \{0, 2, 7\}$, so that (σ^1, σ^2) is an element of $\mathcal{P}_{2\ 5}^{\Upsilon,2}$.

In order to find the exponential generating function for the number of permutations in S_n^m refined by the maximum number of nonoverlapping common Υ-matches, we will introduce a homomorphism on the ring of symmetric functions by defining it on e_n for $n \geq 0$. Let $\vec{q} = (q_1, \ldots, q_m)$. Then set $U_{\vec{q},m}(n) = (-1)^n$ if $n = 0, 1$ and

$$U_{\vec{q},m}(n) = (1-x) \sum_{\omega \in J_\Upsilon^m, ||\omega|| = n} (-1)^{\overline{\ell}(\omega)} \sum_{(\sigma^1,\ldots,\sigma^m) \in \mathcal{P}_\omega^{\Upsilon,m}} \prod_{i=1}^m q_i^{inv(\sigma^i)} \tag{2.16}$$

otherwise. Let $\xi_{\Upsilon,\vec{q},m}$ be the ring homomorphism on Λ with the property that

$$\xi_{\Upsilon,\vec{q},m}(e_n) = \frac{(-1)^n}{\prod_{i=1}^m [n]_{q_i}!} U_{\vec{q},m}(n). \tag{2.17}$$

With these definitions, we can use essentially the same proof as in Theorem 2.15 to prove that

$$\prod_{i=1}^m [n]_{q_i}! \xi_{\Upsilon,\vec{q},m}(h_n) = \sum_{(\sigma^1,\ldots,\sigma^m) \in S_n^m} x^{\Upsilon\text{-}comnlap(\sigma^1,\ldots,\sigma^m)} \prod_{i=1}^m q_i^{inv(\sigma^i)}. \tag{2.18}$$

That is, following the methods set forth in previous proofs, we have

$$\prod_{i=1}^m [n]_{q_i}! \xi_{\Upsilon,,\vec{q},m}(h_n) = \sum_{\lambda \vdash n} \prod_{i=1}^m \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q_i} (-1)^{\ell(\lambda)} B_{\lambda,n} U_{\vec{q},m}(\lambda_1) \cdots U_{\vec{q},m}(\lambda_\ell). \tag{2.19}$$

Start with a brick tabloid of shape (n) and type λ for some $\lambda \vdash n$ to account for the summand and $B_{\lambda,n}$ term. The product $\prod_{i=1}^m \begin{bmatrix} n \\ \lambda \end{bmatrix}_{q_i}$ allow us to associate a subset of the first n integers to each brick m different times so that our brick tabloids will end up with m rows of permutations instead of simply one and, for each cell in the i-th row, we keep track of the q_i raised to the elements in the i-th row which are smaller than the element in the i-th row of that cell which lie in the i-th row of brick to the right of the brick which contain that cell.

The terms of the form $(-1)^{\ell(\lambda)}$ and $U_{\vec{q},m}(\lambda_1) \cdots U_{\vec{q},m}(\lambda_\ell)$ in (2.19) allow us to do the following. First the term $\sum_{\omega \in J_\Upsilon^m, ||\omega||=n} (-1)^{\ell(\omega)} \sum_{(\sigma^1,\ldots,\sigma^m) \in \mathcal{P}_\omega^{\Upsilon,m}} \prod_{i=1}^m q_i^{inv(\sigma^i)}$ in the definition of $U_{\vec{q},m}$ to associate to each brick b of size k, a word $\omega \in J_\Upsilon^m$ with $||\omega|| = k$, an m-tuple of permutations $(\sigma^1, \ldots, \sigma^m) \in \mathcal{P}_\omega^{\Upsilon,m}$ so that we can rearrange the set in the i-th row of a brick b into an ordered sequence $s_{i,b}$ such that $\mathrm{red}(s_{i,b}) = \sigma^i$ and add a weight $q_i^{inv(s_{i,b})}$ for $i = 1, \ldots, m$. It we follows that the i-th row of the brick tabloid will be a permutation γ^i with total weight $q_i^{inv(\gamma^i)}$. The other factors in $U_{\vec{q},m}(\lambda_1)\cdots U_{\vec{q},m}(\lambda_\ell)$ weight the bricks with powers of x and -1 as we did in the proof of Theorem 2.15. After this process of filling brick tabloids, we achieve objects like those found in the proof of Theorem 2.18 except that we have m rows of permutations instead of one. Below we have included such a decorated brick tabloid B when $\Upsilon = \{2\ 1\ 4\ 3\}$ and $m = 2$. Here the total weight of the decorated brick tabloid is $-xq_1^{inv(\sigma^1)}q_2^{inv(\sigma^2)}$ where $\sigma^1 = 7\ 2\ 1\ 4\ 3\ 8\ 6\ 10\ 9\ 5\ 12\ 11$ is the permutation in the top row of the of B and $\sigma^2 = 4\ 3\ 2\ 6\ 5\ 8\ 7\ 10\ 9\ 1\ 12\ 11$ is the permutation in the second row of the of B.

		-1		-1			-1				x
7	2	1	4	3	8	6	10	9	5	12	11
4	3	2	6	5	8	7	10	9	1	12	11

With a slight modification, the involution described in the proof of Theorem 2.15 may be used in this instance. When scanning the bricks in a decorated brick tabloid, scan for situations where Cases 1-4 apply where we use Υ-common matches instead of a τ-matches. This is the natural adaptation for our current situation since we have m permutations in a brick tabloid.

The fixed points under this involution correspond to m different permutations in the symmetric group S_n with powers of x counting the maximum number of nonoverlapping τ-common matches. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{\prod_{i=0}^m [n]_{q_i}!}\sum_{(\sigma^1,\ldots,\sigma^m)\in S_n^m} x^{\Upsilon\text{-}comnlap(\sigma^1,\ldots,\sigma^m)} &= \xi_{\Upsilon,\vec{q},m}\left(\sum_{n=0}^{\infty}h_n t^n\right)\\ &= \xi_{\Upsilon,\vec{q},m}\left(\sum_{n=0}^{\infty}e_n(-t)^n\right)^{-1}.\end{aligned}$$

Then by the same manipulations that we used in the proof of Theorem 2.15, we can prove the following.

THEOREM 2.20. *For any set of permutations $\Upsilon \subseteq S_j$ where $j > 1$,*

$$A^{\Upsilon,\vec{q},m}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\prod_{i=0}^m [n]_{q_i}!}\sum_{(\sigma^1,\ldots,\sigma^m)\in S_n^m:\Upsilon\text{-}commch(\sigma^1,\ldots,\sigma^m)=0}\prod_{i=1}^{n} q_i^{inv(\sigma^i)}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{\prod_{i=1}^m [n]_{q_i}!}\sum_{(\sigma_1,\ldots,\sigma^m)\in S_n^m} x^{\Upsilon\text{-}comnlap(\sigma^1,\ldots,\sigma^m)}\prod_{i=1}^{m} q_i^{inv(\sigma^i)}&\\ = \frac{A^{\Upsilon,\vec{q},m}(t)}{(1-x)+x(1-t)A^{\Upsilon,\vec{q},m}(t)}&.\end{aligned}$$

Let $J_0(t) = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{2^{2n}(n!)^2}$ be the Bessel function of the first kind. The generating function for the maximum number of nonoverlapping common descents appearing in two permutations—that is, we are taking τ to be the permutation 2 1 and $m = 2$ in the statement of 2.20—is equal to

$$\frac{1}{x(1-t)+(1-x)J_0(2\sqrt{t})}. \tag{2.20}$$

Similarly by taking $\Upsilon = \{2\ 1\ 3, 3\ 1\ 2\}$, one may show that the generating function for the maximum number of nonoverlapping common valleys in two permutations is

$$\frac{3 + J_0\left(4\sqrt{-t}\right)}{4 - x - 3tx + x(1-t)J_0\left(4\sqrt{-t}\right)}. \tag{2.21}$$

The goal for the end of this section is to show that, under certain conditions, we are able to take $A(t)$ and refine permutations in the symmetric group by the statistic τ-mch. The methods to find these generating functions are trickier than the situation of nonoverlapping τ-matches in the sense that we will the symmetric function $p_{n,\nu}$ as opposed to h_n. The involution we will need to give fixed points corresponding to permutations, however, will be easier than those in the nonoverlapping case.

The only case for which we can apply our forthcoming methods is when I_τ has only one element which is at least half the size of τ. That is, if $\tau \in S_j$, then we will assume that $I_\tau = \{k_\tau\}$ where $2k_\tau \geq j$.

In order to achieve our goal, we will break $A(t)$ into k_τ pieces. For $0 \leq m < k_\tau$, let $A_m(t) = \sum_{n=1}^{\infty} t^{nk_\tau - m}\left(A(t)|_{t^{nk_\tau - m}}\right)$ so that $A(t) = 1 + \sum_{m=0}^{k_\tau - 1} A_m(t)$ where we are denoting $\cdot|_{t^k}$ as the coefficient of t^k in $\cdot$. One may immediately find $A_m(t)$ for $m \geq 1$ from $A(t)$ by the relationship

$$A_m(t) = \frac{1}{k_\tau}\sum_{\ell=0}^{k_\tau - 1} \mathbf{e}^{\frac{2\pi i}{k_\tau}\ell m} A\left(\mathbf{e}^{\frac{2\pi i}{k_\tau}\ell} t\right)$$

which is nothing more than an elaborate exploitation of the fact that the sum of all ${k_\tau}^{\text{th}}$ roots of unity is equal to zero. For $m = 0$, $A_0(t) = -1 + \frac{1}{k_\tau}\sum_{\ell=0}^{k_\tau - 1} A\left(\mathbf{e}^{\frac{2\pi i}{k_\tau}\ell} t\right)$.

Let ϑ be the homomorphism defined on e_n such that

$$\vartheta(e_n) = \begin{cases} 1 & \text{if } n = 0, \text{ and} \\ (-1)^n(-x)(1-x)^{n-1}\ A(t)|_{t^{nk_\tau}} & \text{if } n \geq 1. \end{cases}$$

Let $0 \leq m < k_\tau$. As indicated in the introduction to this section, we will apply ϑ on the our recursively defined symmetric function with parameter a function. This function, which will help to weight the last brick in a brick tabloid differently than the other bricks, is υ defined by

$$\upsilon(n) = \frac{A(t)|_{t^{nk_\tau - m}}}{A(t)|_{t^{nk_\tau}}}.$$

The homomorphism ϑ and the function υ will be used in tandem in the proof of Theorem 2.21.

THEOREM 2.21. *For $\tau \in S_j$, if $I_\tau = \{k_\tau\}$ where $2k_\tau \geq j$, then*

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-}mch(\sigma)} = \frac{\sum_{m=0}^{k_\tau - 1}(1-x)^{m/k_\tau} A_m\left(t\ \sqrt[k_\tau]{1-x}\right)}{(1-x) - xA_0\left(t\ \sqrt[k_\tau]{1-x}\right)}.$$

Proof. First we will expand $(nk_\tau - m)!p_{n,\upsilon}$ in terms of the elementary basis. We have $(nk_\tau - m)!\vartheta(p_{n,\upsilon})$ is equal to

$$\begin{aligned}(nk_\tau - m)! &\sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_\upsilon(B_{\lambda,n})\vartheta(e_\lambda)\\ &= (nk_\tau - m)! \sum_{\lambda \vdash n} w_\upsilon(B_{\lambda,(n)}) x^{\ell(\lambda)}(1-x)^{n-\ell(\lambda)} \left(A(t)|_{t^{k_\tau \lambda_1}} \right) \cdots \left(A(t)|_{t^{k_\tau \lambda_\ell}} \right).\end{aligned}$$

By the definition of the weight on a brick tabloid, this may be written to read

$$(nk_\tau - m)! \sum_{\lambda \vdash n} \sum_{\substack{\text{brick tabloids of shape} \\ \lambda \text{ with bricks } b_1, \ldots, b_\ell}} x^\ell (1-x)^{n-\ell} \left(A(t)|_{t^{k_\tau b_1}} \right) \cdots \left(A(t)|_{t^{k_\tau b_\ell - m}} \right),$$

which, since $A(t)$ is the exponential generating function for the number of permutations in S_n with τ-mch$(\sigma) = 0$, is equal to

$$\begin{aligned}(2.22) \quad &\sum_{\lambda \vdash n} \sum_{\substack{\text{brick tabloids of shape} \\ \lambda \text{ with bricks } b_1, \ldots, b_\ell}} \binom{nk_\tau - m}{k_\tau b_1, \cdots, k_\tau b_{\ell-1}, k_\tau b_\ell - m} x^\ell (1-x)^{n-\ell}\\ &\times |\{\sigma \in S_{k_\tau b_1} : \tau\text{-mch}(\sigma) = 0\}| \cdots |\{\sigma \in S_{k_\tau b_\ell - m} : \tau\text{-mch}(\sigma) = 0\}|.\end{aligned}$$

Start with a brick tabloid of shape (n) and type λ with bricks $b_1, \ldots, b_\ell$ to take care of the two summands. Multiply the length of every brick by k_τ. The factor of the form $x^\ell(1-x)^{n-\ell}$ allows us to place a x in the final cell of each brick and also allows us to place a choice of either 1 or $-x$ in every ${k_\tau}^{\text{th}}$ cells in each brick.

The multinomial coefficient in (2.22) chooses which subset of the first $nk_\tau - m$ integers are to be placed in each brick. Every cell in every brick—except for the last brick—will be filled with an integer. The last brick of length b_ℓ will be filled with $k_\tau b_\ell - m$ integers. Use the factor of $|\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}|$ we have for each brick to arrange the integers into a permutation without any τ-matches. Forming brick tabloids in this way uses every term in (2.22). One such object when taking $\tau = 1\ 3\ 2$ (and therefore $k_\tau = 2$), $n = 12$, and $m = 1$ is found below.

	1		x		x		x		$-x$		x
10	9	2	4	1	11	5	3	8	6	7	

Define the weight of such a decorated brick tabloid to be the product of the x, $-x$, and 1's. We will now describe a sign-reversing, x-weight preserving involution to rid ourselves of any brick tabloid with negative total weight. Scan the bricks from left to right looking for one of the following two situations:

(1) a cell containing $-x$, or
(2) two consecutive bricks which may be combined to form a permutation without a τ-match.

If situation 1 appears first, break the brick containing the $-x$ into two bricks immediately after the offending $-x$ and reverse the sign on the $-x$ to x. This places us into situation 2. Thus, if we find ourselves in situation 2, combine the two bricks and reverse the sign on the x now in the middle. It is not difficult to see that this is a sign-reversing, x-weight involution. As an example of this involution we have included the image of the brick tabloid depicted earlier in this proof.

	1		$-x$		x		x		$-x$		x
10	9	2	4	1	11	5	3	8	6	7	

A fixed point under this involution cannot have any powers of $-x$ in the tabloid and we cannot combine any two bricks in the tabloid to create a permutation without a τ-match. This means that there must be a τ-match between every two bricks and that the x weight on a fixed point under this involution is one power greater than the number of τ-matches (this extra power of x comes from the terminal brick). The condition that $2k_\tau \geq j$ in the hypothesis of this theorem is invoked here to ensure that the powers of x on a fixed point actually correspond to the number of τ-matches. Otherwise, if this condition was not met, then there could possibly be overlapping τ-matches not corresponding to a power of x. One such fixed point is shown below.

	1		1		x		x		1		x
10	9	2	4	1	11	5	3	8	6	7	

We have just explained why $(nk_\tau - m)!\vartheta(p_{n,\upsilon})$ is equal to the sum of $x^{\tau\text{-mch}(\sigma)}$ for all $\sigma \in S_{nk_\tau - m}$. Summing over all values of m,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} &= \sum_{m=0}^{k_\tau - 1} \sum_{n=0}^{\infty} \frac{t^{nk_\tau - m}}{(nk_\tau - m)!} \sum_{\sigma \in S_{nk_\tau - m}} x^{\tau\text{-mch}(\sigma)} \\ &= \sum_{m=0}^{k_\tau - 1} \frac{1}{xt^m} \vartheta \left(\sum_{n=1}^{\infty} p_{n,\upsilon} t^{nk_\tau} \right).\end{aligned}$$

Using (1.12), this is equal to

$$\sum_{m=0}^{k_\tau - 1} \frac{1}{xt^m} \frac{\sum_{n=1}^{\infty} (-1)^{n-1} \upsilon(n) \vartheta(e_n) t^n}{\sum_{n=0}^{\infty} (-1)^n \vartheta(e_n) t^n} = \sum_{m=0}^{k_\tau - 1} \frac{\sum_{n=1}^{\infty} (1-x)^{n-1} t^{nk_\tau - m} \left(A(t)|_{t^{nk_\tau - m}} \right)}{1 - x \sum_{n=1}^{\infty} (1-x)^{n-1} t^{nk_\tau} \left(A(t)|_{t^{nk_\tau}} \right)}$$

which in turn is equal to the desired expression. □

As a first example of the use of Theorem 2.21, let us take $\tau = 2\ 1$ so that a τ-match in a permutation is actually a descent. The hypotheses of the theorem hold since $k_\tau = 1$ and $j = 2$. Since $k_\tau = 1$ in this situation, there is no need to break the function $A(t)$ into pieces. The exponential generating function $A_0(t)$ for the number of permutations without a descent is $\mathbf{e}^t - 1$. Theorem 2.21 gives the well known formula that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} = 1 + \frac{\mathbf{e}^{t(1-x)} - 1}{(1-x) - x\left(\mathbf{e}^{t(1-x)} - 1\right)} = \frac{x-1}{x - \mathbf{e}^{t(x-1)}}.$$

To provide a more involved example where we must break up the function $A(t)$, let us consider $\tau = 1\ 3\ 2$. In this case, $k_\tau = 2$ while $j = 3$ so the hypotheses of the theorem hold. By (2.3),

$$A(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} |\{\sigma \in S_n : \tau\text{-mch}(\sigma) = 0\}| = \left(1 - \int \mathbf{e}^{-t^2/2}\, dt \right)^{-1}.$$

Taking $A_0(t) = \frac{A(t)+A(-t)}{2} - 1$ and $A_1(t) = \frac{A(t)-A(-t)}{2}$ and plugging these generating functions into the statement of Theorem 2.21, simplification gives that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} = \left(1 - \int \mathbf{e}^{t^2(x-1)/2}\, dt\right)^{-1}$$

in the case $\tau = 1\ 3\ 2$.

There was nothing other than ease of explanation in the proof of Theorem 2.21 which prevented us from stating this result for collections of permutations Υ provided that $I_\Upsilon = \{k_\tau\}$ where $2k_\tau \geq j$. For example, when taking $\Upsilon = \{3\ 1\ 2, 2\ 1\ 3\}$, we may find information about distribution of valleys over the symmetric group. Since the number of permutations of n without any valleys is 2^{n-1}, we may deduce that $A_0(t) = \sinh^2(t)$ and $A_1(t) = \sinh(t)\cosh(t)$. Theorem 2.21 gives that the generating function refining the symmetric group by the total number of valleys is equal to

$$\frac{\sinh^2\left(t\sqrt{1-x}\right) + \sqrt{1-x}\sinh\left(t\sqrt{1-x}\right)\cosh\left(t\sqrt{1-x}\right)}{1 - x - x\sinh^2\left(t\sqrt{1-x}\right)} = \frac{1}{\sqrt{x-1}\cot\left(t\sqrt{x-1}\right) - 1}.$$

The main theorem in Section 2.3 was extended to the case of inversions and multiple permutations and so may Theorem 2.21. As for inversions, if we take $A_q(t)$ as in the statement of 2.19 and split $A_q(t)$ into k_τ pieces by letting $A_{q,m}(t) = \sum_{n=1}^{\infty} t^{nk_\tau - m}\left(A_q(t)|_{t^{nk_\tau - m}}\right)$ so that $A_q(t) = 1 + \sum_{m=0}^{k_\tau - 1} A_{q,m}(t)$, Theorem 2.21 may be written to read that if $I_\tau = \{k_\tau\}$ where $2k_\tau \geq j$, then

$$\sum_{n=1}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\tau\text{-mch}(\sigma)} q^{\mathrm{inv}(\sigma)} = \frac{\sum_{m=0}^{k_\tau - 1} (1-x)^{m/k_\tau} A_{q,m}\left(t \sqrt[k_\tau]{1-x}\right)}{(1-x) - xA_{q,0}\left(t \sqrt[k_\tau]{1-x}\right)}.$$

The justification for the above equality is found by combining the q labeling ideas in the proof of 2.19 with the labeling techniques in the proof of Theorem 2.21. This is perfectly valid since the involution in the proof of Theorem 2.21 does not rearrange any of the integers in the permutation and thus any q count dependent on the placement of integers in a permutation is preserved.

As an example of the use of this q-analogue, if we let $\mathrm{Exp}_q(t(x-1))$ be q-analogue of the exponential function defined by $\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} q^{\binom{n}{2}}$, then (2.3) becomes

$$\sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{inv}(\sigma)} = \frac{x-1}{x - \mathrm{Exp}_q(t(x-1))}.$$

In addition to the case of an inversion count, Theorem 2.21 may be written in terms of multiples of permutations. In the case of pairs of permutations, let $A_2(t)$ to be equal to

$$\sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} |\{(\sigma, \pi) \in S_n \times S_n : \tau\text{-}commch(\sigma, \pi) = 0\}|$$

and break $A_2(t)$ into k_τ pieces by defining $A_{2,m}(t) = \sum_{n=1}^{\infty} t^{nk_\tau - m} \left(A_2(t)|_{t^{nk_\tau - m}} \right)$. Then, Theorem 2.21 may be specialized to read

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\sigma,\pi \in S_n} x^{\tau\text{-commch}(\sigma,\pi)} = \frac{\sum_{m=0}^{k_\tau - 1} (1-x)^{m/k_\tau} A_{m,2}\left(t \sqrt[k_\tau]{1-x}\right)}{(1-x) - xA_{0,2}\left(t \sqrt[k_\tau]{1-x}\right)}.$$

The proof of this equality is a straightforward combination of the labeling ideas in 2.20 with the proof of Theorem 2.21.

From (2.20), the function $A_2(t)$ for the number of pairs of permutations without any common descents is $1/J_0\left(2\sqrt{t}\right)$. By the above remarks, the number of common descents in two permutations

$$\frac{1-x}{J_0\left(2\sqrt{t(1-x)}\right) - x}$$

Furthermore, looking at (2.21), the function $A_2(t)$ without any common valleys is $\left(3 + J_0\left(4\sqrt{-t}\right)\right)/4$. Therefore, the generating function registering the number of common valleys in two permutations is

$$\frac{6x - 6 + \left(x - 1 - \sqrt{1-x}\right) J_0\left(4i\sqrt{t\sqrt{1-x}}\right) + \left(x - 1 + \sqrt{1-x}\right) J_0\left(4\sqrt{t\sqrt{1-x}}\right)}{-8 + 6x + xJ_0\left(4\sqrt{-t\sqrt{1-x}}\right) + xJ_0\left(4\sqrt{t\sqrt{1-x}}\right)}$$

The last two ideas we have included in this section involved adding powers of q counting inversions into the brick tabloids and adding multiple rows of permutations in the brick tabloids. There is nothing which prevents us from combining these ideas. That is, by taking $A_{\Upsilon,\vec{q},m}(t)$ as in Theorem 2.20, we can find a generating function for τ-commch$(\sigma^1, \ldots, \sigma^m)$ together with $q_1^{\text{inv}(\sigma_1)} \cdots q_m^{\text{inv}(\sigma_m)}$.

2.4. Descents, major indices, and inversions

Ira Gessel gave a generating function for

$$\sum_{\sigma \in S_n} x^{\text{des}(\sigma)} u^{\text{maj}(\sigma)} q^{\text{inv}(\sigma)} \tag{2.23}$$

both in his thesis and in a paper co-authored with Adriano Garsia [**42, 40**]. We will begin this section by reproving this result. Then, we will show how similar proofs indicate a systematic approach to finding more generating functions for similar permutation statistics on other collections of objects.

Define a ring homomorphism ξ_7 by defining it on the elementary symmetric function e_n such that

$$\xi_7(e_n) = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0 + \cdots + i_k = n}} \frac{(u/v)^{0i_0 + \cdots + ki_k}}{[i_0]_{p,q}! \cdots [i_k]_{p,q}!} q^{\binom{i_0}{2} + \cdots + \binom{i_k}{2}}.$$

Understanding the action of ξ_7 on h_n is at the heart of our proof of Theorem 2.22 below.

THEOREM 2.22.

$$\sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!(y,x;v,u)_{n+1}} \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} u^{\mathrm{maj}(\sigma)} v^{\mathrm{comaj}(\sigma)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}$$

$$= \sum_{k\geq 0} \frac{x^k}{y^{k+1}\mathbf{e}_{p,q}^{-t(u/v)^0}\cdots \mathbf{e}_{p,q}^{-t(u/v)^k}}.$$

PROOF. First we apply ξ_7 to $[n]_{p,q}!h_n$. We have $[n]_{p,q}!\xi_7(h_n)$ is equal to

$$[n]_{p,q}! \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \xi_7(e_\lambda)$$

$$= [n]_{p,q}! \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{j=1}^{\ell(\lambda)} \sum_{\substack{i_{j,0},\ldots,i_{j,k}\geq 0\\ i_{j,0}+\cdots+i_{j,k}=\lambda_j}} \frac{(u/v)^{0i_{j,0}+\cdots+ki_{j,k}}}{[i_{j,0}]_{p,q}!\cdots[i_{j,k}]_{p,q}!} q^{\binom{i_{j,0}}{2}+\cdots+\binom{i_{j,k}}{2}}.$$

Rewriting p,q-analogues, the last equation is equal to

$$\sum_{\lambda\vdash n} \begin{bmatrix} n\\ \lambda\end{bmatrix}_{p,q} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{j=1}^{\ell(\lambda)} \sum_{\substack{i_{j,0},\ldots,i_{j,k}\geq 0\\ i_{j,0}+\cdots+i_{j,k}=\lambda_j}} \begin{bmatrix} \lambda_j\\ i_{j,0},\ldots,i_{j,k}\end{bmatrix}_{p,q} \times (u/v)^{0i_{j,0}+\cdots+ki_{j,k}} q^{\binom{i_{j,0}}{2}+\cdots+\binom{i_{j,k}}{2}}. \tag{2.24}$$

Let us begin to build combinatorial objects from (2.24) with a brick tabloid T. In each nonterminal cell in each brick, place a -1. In each terminal cell in a brick, place a 1. This uses the summand, $B_{\lambda,n}$ term, and $(-1)^{n-\ell(\lambda)}$ terms in (2.24). For each brick in T, choose nonnegative integers $i_0,\ldots,i_k$ that sum to the total length of the brick. This accounts for the product and second sum in (2.24). Using powers of (u/v), these choices for $i_0,\ldots,i_k$ can be recorded in T. In each brick, place a power of (u/v) in each cell such that the powers weakly increase from left to right and the number of occurrences of $(u/v)^j$ is i_j. At this point, we have constructed and object which may look something like the object below.

-1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
u^1v^{-1}	u^1v^{-1}	u^3v^{-3}	u^0v^0	u^0v^0	u^0v^0	u^0v^0	u^2v^{-2}	u^3v^{-3}	u^3v^{-3}	u^1v^{-1}	u^1v^{-1}

The only components in (2.24) which have not been used involve powers of p and q. We will explain how these powers of p and q will fill the cells of T with a permutation of n such that a decrease must occur between consecutive cells labeled with the same power of (u/v). Along with this permutation of n, powers of p and q will be placed in each cell recording coinversions and inversions.

By Lemma 1.24, the $\begin{bmatrix} n\\ \lambda\end{bmatrix}_{p,q}$ term in (2.24) gives powers of q and p counting inversions and coinversions of a rearrangement of λ_1 0's, λ_2 1's, etc. We will use this to select which integers in a permutation of n will appear in each brick. Suppose that the size of the bricks read from left to right in T are $b_0,\ldots,b_k$. Consider a rearrangement r of $0^{b_0},\ldots,k^{b_k}$ and construct a permutation σ_r by labeling the 0's from left to right with $1,2,\ldots,b_0$, the 1's from right to left with $b_0+1,\ldots,b_0+b_1$, and in general the i's from right to left with $1+\sum_{j=1}^{i-1} b_j, \ldots, b_i+\sum_{j=1}^{i-1} b_j$. In this way, σ_r^{-1} starts with the positions of the 0's in r increasing order, followed by the positions of the 1's in r in increasing order, etc. For example, if T has bricks of length $2,7,3$, one possible rearrangement to consider is $r = 1\ 0\ 1\ 1\ 1\ 2\ 0\ 1\ 2\ 1\ 0\ 1\ 1$. Below we display σ_r and σ_r^{-1}.

		1	2	3	4	5	6	7	8	9	10	11	12
r	$=$	1	0	1	1	1	0	1	2	1	0	1	1
σ_r	$=$	4	1	5	6	11	2	7	12	8	3	9	10
σ_r^{-1}	$=$	2	6	10	1	3	4	7	9	11	12	5	8.

This tells us that when selecting a permutation of 12 to place in T, the integers $2,6,10$ should appear in the brick of size 3, the integers $1,3,4,7,9,11,12$ should appear in the brick of size 7, and the integers $5,8$ should appear in the brick of size 2. It is not difficult to verify that $\mathrm{inv}(r) = \mathrm{inv}(\sigma_r) = \mathrm{inv}(\sigma_r^{-1})$.

For each brick of length λ_j, there is an unused term of the form

$$\begin{bmatrix} \lambda_j \\ i_0, \ldots, i_k \end{bmatrix}_{p,q} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}}$$

where $i_0 + \cdots + i_k = \lambda_j$. Lemma 1.24 enables us to start with a rearrangement a of i_0 0's, i_1 1's, etc. to use the p,q-multinomial coefficient. Record from right to left the 0's in a with $1, \ldots, i_0$. Then record the 1's in a from right to left with $i_0+1, \ldots, i_1$. Continue this process k times to form a permutation of λ_j from a, τ_a^{-1}. The inverse, τ_a, records the places of the 0's, 1's, etc., and therefore must have decreasing sequences of length $i_0, \ldots, i_k$. Let $\overline{\tau_a}$ be the permutation τ_a where the integers $1, \ldots, \lambda_j$ have been replaced with whatever integers the factor $\left[\begin{smallmatrix} n \\ \lambda \end{smallmatrix}\right]_{p,q}$ dictates should appear in the j^{th} brick.

For example, if $k = 3$, $i_0 = 4$, $i_1 = 0$, $i_2 = 1$, and $i_3 = 2$, a permutation of 7 may be formed from $0\ 2\ 0\ 3\ 3\ 0\ 0$. Continuing our example from above, the brick of size 7 should contain the integers $1,3,4,7,9,11$, and 12. The permutations τ_a^{-1}, τ_a, and $\overline{\tau_a}$ can be found:

		1	2	3	4	5	6	7
a	$=$	0	2	0	3	3	0	0
τ_a^{-1}	$=$	4	5	3	7	6	2	1
τ_a	$=$	7	6	3	1	2	5	4
$\overline{\tau_a}$	$=$	12	11	4	1	3	9	7.

By construction, we have that

$$\mathrm{inv}(\overline{\tau_a}) = \mathrm{inv}(\tau_a) = \mathrm{inv}(\tau_a^{-1}) = \mathrm{inv}(r) + \binom{i_0}{2} + \cdots + \binom{i_k}{2}.$$

Therefore, for each brick of size λ_j, we may associate a permutation of λ_j such that the permutation must have a descent if two consecutive cells have the same power of (u/v). By taking along a power of $q^{\mathrm{inv}(\overline{\tau_a})}$ and $p^{\mathrm{coinv}(\overline{\tau_a})}$, we are able to account for the factors in (2.24) of the form $\left[\begin{smallmatrix} \lambda_j \\ i_0,\ldots,i_k \end{smallmatrix}\right]_{p,q} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}}$. Every term in (2.24) has now been used.

Below we give one example of such an object created in this manner.

-1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
u^1v^{-1}	u^1v^{-1}	u^3v^{-3}	u^0v^0	u^0v^0	u^0v^0	u^0v^0	u^2v^{-2}	u^3v^{-3}	u^3v^{-3}	u^1v^{-1}	u^1v^{-1}
q^9p^2	q^1p^9	q^4p^5	q^8p^0	q^7p^0	q^2p^4	q^0p^5	q^0p^4	q^3p^0	q^1p^1	q^1p^0	q^0p^0
10	2	6	12	11	4	1	3	9	7	8	5

These decorated brick tabloids of shape (n) and type λ for some $\lambda \vdash n$ have the following properties: the cells in each brick contain -1 except for the final cell which contains 1, each cell contains a power of (u/v) such that the powers weakly increase within each brick, T contains a permutation of n which must have a decrease between consecutive cells within a brick if the cells are marked with the same power of (u/v), and each cell contains a power of q counting the number of cells to the right which are smaller and a power of p counting the number of cells to the right which are larger.

The weight of such an object is defined to be the product of all (u/v), p, q, and -1 labels. In this way, $[n]_{p,q}!\xi_7(h_n)$ is the weighted sum over all possible decorated brick tabloids. A sign-reversing weight-preserving involution will rid ourselves of any T with a negative weight.

Scan the cells from left to right looking for either a cell containing -1 or two consecutive bricks which may be combined to preserve the properties of these collection of objects. If a -1 is scanned first, break the brick containing the -1 into two immediately after the violation and change the -1 to 1. If the second situation is scanned first, glue the bricks together and change the 1 in the middle to -1. The image of the previous object is displayed below.

1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
u^1v^{-1}	u^1v^{-1}	u^3v^{-3}	u^0v^0	u^0v^0	u^0v^0	u^0v^0	u^2v^{-2}	u^3v^{-3}	u^3v^{-3}	u^1v^{-1}	u^1v^{-1}
q^9p^2	q^1p^9	q^4p^5	q^8p^0	q^7p^0	q^2p^4	q^0p^5	q^0p^4	q^3p^0	q^1p^1	q^1p^0	q^0p^0
10	2	6	12	11	4	1	3	9	7	8	5

By definition, this is a sign-reversing weight-preserving involution with fixed points such that: there are no bricks containing -1 and therefore every brick must be of length 1, the powers of (u/v) weakly decrease, and if two consecutive bricks have the same power (u/v), then the permutation must increase there. One example of a fixed point may be found below.

1	1	1	1	1	1	1	1	1	1	1	1
u^3v^{-3}	u^3v^{-3}	u^3v^{-3}	u^2v^{-2}	u^2v^{-2}	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^0v^0	u^0v^0
q^3p^8	q^4p^6	q^5p^4	q^1p^7	q^1p^6	q^1p^5	q^1p^4	q^1p^3	q^2p^1	q^2p^0	q^0p^1	q^0p^0
4	6	8	2	3	5	7	9	11	12	1	10

We now turn our attention to counting fixed points.

Suppose that the powers of (u/v) in a fixed point are $r_1, \ldots, r_n$ when read from left to right. It must be the case that $k \geq r_1 \geq \cdots \geq r_n$. Define nonnegative integers a_i by $a_i = r_i - r_{i+1}$ for $i = 1, \ldots, n-1$ and let $a_n = r_n$. It follows that $r_1 + \cdots + r_n = a_1 + 2a_2 + \cdots + na_n$, $a_1 + \cdots + a_n = r_1 \leq k$, and if σ is the permutation in a fixed point, $a_i \geq \chi(\sigma_i > \sigma_{i+1})$. In this way, the weighted over all fixed points—and therefore $[n]_{p,q}!\xi_7(h_n)$—is equal to

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} \sum_{\substack{a_1+\cdots+a_n \leq k \\ a_i \geq \chi(\sigma_i > \sigma_{i+1})}} (u/v)^{a_1+2a_2+\cdots+na_n} = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}$$
$$\times \sum_{a_1 \geq \chi(\sigma_1 > \sigma_2)} \cdots \sum_{a_n \geq \chi(\sigma_n > n+1)} (x/y)^{a_1+\cdots+a_n} (u/v)^{a_1+2a_2+\cdots+na_n} \Bigg|_{(x/y)^{\leq k}}.$$

Where $expression|_{t^{\leq k}}$ means to sum the coefficients of t^j for $j = 0, \ldots, k$ in *expression*. Rewriting the above equation, we have

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} \frac{((x/y)(u/v))^{\chi(\sigma_1 > \sigma_2)} \cdots ((x/y)(u/v)^n)^{\chi(\sigma_n > n+1)}}{(1-(x/y)(u/v)) \cdots (1-(x/y)(u/v)^n)} \Bigg|_{(x/y)^{\leq k}}$$
$$= \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} \frac{(x/y)^{\mathrm{des}(\sigma)} (u/v)^{\mathrm{maj}(\sigma)}}{(1-(x/y)(u/v)) \cdots (1-(x/y)(u/v)^n)} \Bigg|_{(x/y)^{\leq k}}.$$

Dividing by $(1 - x/y)$ and factoring the powers of y and v out of the denominator allows the above expression to be rewritten as

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} \frac{x^{\mathrm{des}(\sigma)} y^{n-\mathrm{des}(\sigma)+1} u^{\mathrm{maj}(\sigma)} v^{\binom{n+1}{2}-\mathrm{maj}(\sigma)}}{(y-x) \cdots (yv^n - xu^n)} \Bigg|_{(x/y)^k}.$$

Where $expression|_{t^j}$ means to select the coefficient of t^j in *expression*. Therefore, we have

$$\sum_{n \geq 0} \frac{yt^n}{[n]_{p,q}!(y,x;v,u)_{n+1}} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} u^{\mathrm{maj}(\sigma)} v^{\mathrm{comaj}(\sigma)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}$$
$$= \sum_{k \geq 0} (x/y)^k \xi_7 \left(\sum_{n \geq 0} t^n h_n \right)$$
$$= \sum_{k \geq 0} (x/y)^k \left(\sum_{n \geq 0} (-t)^n \xi_7(e_n) \right)^{-1}$$
$$= \sum_{k \geq 0} (x/y)^k \left(\sum_{n \geq 0} (-t)^n \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k = n}} \frac{(u/v)^{0i_0+\cdots+ki_k}}{[i_0]_{p,q}! \cdots [i_k]_{p,q}!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} \right)^{-1},$$

which may be simplified to equal the desired expression. ☐

Theorem 2.22 was proved in a paper of Garsia and Gessel by playing with the set of non-negative integer valued functions on the first n integers [**40**]. Without their work, the homomorphism ξ_7 would have been more difficult to find.

CHAPTER 3

Generating functions for other objects

3.1. Wreath product statistics

Let G be a finite group. Recall that the group $G \wr S_n$ is defined as

$$G \wr S_n = \{(f, \sigma) \,|\, f : \{1, \ldots, n\} \to G \text{ and } \sigma \in S_n\}$$

and is referred to as the wreath product of G with S_n. In Section 1.4 we show how to multiply two elements of the wreath product groups and give a few different ways to represent elements in $G \wr S_n$.

In this section, we show how different weighting functions can give permutation statistics for wreath product groups. Close attention will be paid to the hyperoctahedral group B_n and its subgroup D_n. The hyperoctahedral group B_n is isomorphic to $\{-1, 1\} \wr S_n$ while the subgroup D_n consists of those elements in B_n with the property that there are an even number of -1's in the image of f.

Let g be a function mapping a finite set S to some set of indeterminates $\{a_1, \ldots, a_c\}$. Form an equivalence relation on S by defining $s_1 \sim_g s_2$ if and only if $g(s_1) = g(s_2)$. (For wreath product groups, we will take S to be the group G.) For $1 \leq i \leq c$ and $s = (s_1, \ldots, s_n) \in S \times \cdots \times S$, let $n_i(s)$ be the statistic counting the number of indices j such that $g(s_j) = a_i$. For example, suppose $S = \{1, 2, 3, 4\}$ and $g : S \to \{a_1, a_2\}$ such that $g(1) = g(2) = a_1$ and $g(3) = g(4) = a_2$. Then

$$n_1((3, 4, 2, 4, 1, 3)) = 2 \qquad \text{and} \qquad n_2((3, 4, 2, 4, 1, 3)) = 4.$$

The statistics n_i when applied to wreath product groups only give information about the group element paired with an integer in a permutation, not the integer itself. A new statistic can take more of the group structure of $G \wr S_n$ into account. In order to do this, we define a partial order on $S \times \{1, \ldots, n\}$ for a finite set S. Let $s, s' \in S$ and $i, j \in \{1, \ldots, n\}$. Define Ω such that $(s, i) <_\Omega (s', j)$ if $i < j$ and s, s' are in the same equivalence class given by $\sim_g$.

Using this partial order, permutation statistics analogous to those of the symmetric group may be defined. Let σ be a permutation written in one line notation where each integer is paired with an element from S. Let σ_i be the i^{th} integer and corresponding element from S. For example, if $\sigma = (s_2, 2)\ (s_3, 3)\ (s_1, 1)$, then $\sigma_1 = (s_2, 2), \sigma_2 = (s_3, 3)$, and $\sigma_3 = (s_1, 1)$. In the case of letting S be a finite group G, σ may be thought of as an element of the wreath product $G \wr S_n$.

Define

$$\operatorname{cdes}(\sigma) = \sum_i \chi(\sigma_{i+1} <_\Omega \sigma_i).$$

The statistic cdes counts descents of integers paired with elements from S in the same equivalence class. For example, by using the convention that $(+1, i) = i$ and $(-1, i) = -i$ when considering elements in $\mathbb{Z}_2 \times \{1, \ldots, n\}$, if $\sigma = –3\ \ –2\ \ +5\ \ –1\ \ +6\ \ +4 \in B_6$ (where $B_n = \mathbb{Z}_2 \wr S_n$), then $\operatorname{cdes}(\sigma) = 2$. By taking the set S to be a set with only one element, cdes is equal to the classic descent statistic applied to the symmetric group. Define the statistic $\mathrm{cris}(\sigma)$ to such that $\mathrm{cris}(\sigma)+\mathrm{cdes}(\sigma)=n$ so that $\mathrm{cris}(\sigma)$ is the natural companion to $\mathrm{cdes}(\sigma)$. (The "c" in the front of des and ris stands for "class" because these statistics only register when consecutive integers are in the same equivalence class.)

To register these statistics over the group $G \wr S_n$, we define a weighting function in the variables x, y and $A_1, \ldots, A_c$ where $A_j = |C_j| a_j$ and $|C_j|$ is the size of the equivalence class containing the elements in S mapping to a_j. Let f_{11} be the weighting function defined such that for $n \geq 1$,

$$f_{11}(n) = -y(x-y)^{n-1}\left((A_1)^n + \cdots + (A_c)^n\right).$$

The following theorem is not stated in the most general way possible; we are taking the case where a finite group G takes the role of S.

THEOREM 3.1.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in G \wr S_n} a_1^{n_1(\sigma)} \cdots a_c^{n_c(\sigma)} x^{\mathrm{cdes}(\sigma)} y^{\mathrm{cris}(\sigma)} = \frac{x-y}{x + y\left(c - 1 - \sum_{i=1}^{c} \mathbf{e}^{A_i t(x-y)}\right)}.$$

PROOF. The weighting function f_{11} allows us to label each cell in a brick of size n in a brick tabloid with A_i for some $i = 1, \ldots, c$ in addition to the usual x and y labels. This follows because the $((A_1)^n + \cdots + (A_c)^n)$ term in the definition of f_{11} allows us to have one choice of A_i^n for every brick of size n along with the factor of $(-y)(x-y)^{n-1}$ labels bricks in the same way as the weighting function f_1. An example of one such $T \in \mathcal{T}_{f_{11}}$ may be found below.

x	x	y	$-y$	x	$-y$	y	x	$-y$	y	$-y$	y
A_1	A_1	A_1	A_2	A_2	A_2	A_2	A_1	A_1	A_1	A_1	A_1
5	4	1	8	6	3	2	12	9	7	11	10

Apply the following involution. Scan the bricks from left to right for either a brick with a $-y$ label or two consecutive bricks where the integer labels decrease and the two consecutive bricks are labeled with the same A_i. If a $-y$ label is found, break the brick into two and reverse the sign on the $-y$. If two consecutive bricks where the integer labels decrease and the two bricks have the same A_i label, combine the bricks. This process is a sign-reversing weight-preserving involution.

The fixed points under this involution correspond to elements in $G \wr S_n$ where the sum over the weights of all fixed points is the sum in the statement of the theorem because $A_j = |C_j| a_j$ and because there are $|C_j|$ choices for $\sigma \in G$ such that $g(\sigma) = a_j$. The proof of this theorem is complete by an application of routine symbolic manipulations. □

Specializations of Theorem 3.1 can give information about the hyperoctahedral group B_n and its subgroup D_n. Define a function $g : \{1, -1\} \to \{u, v\}$ such that $g(1) = u$ and $g(-1) = v$. In this way, for any signed permutation $\sigma \in B_n$, $n_1(\sigma)$ counts the number of positive signs in σ and $n_2(\sigma)$ counts the number of negative signs in σ. To employ more suggestive notation, let $\mathrm{pos}(\sigma)$ be the number of positive signs and $\mathrm{neg}(\sigma)$ be the number of negative signs in $\sigma \in B_n$. Then, a particular case of Theorem 3.1 shows

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in B_n}u^{\mathrm{pos}(\sigma)}v^{\mathrm{neg}(\sigma)}x^{\mathrm{cdes}(\sigma)}y^{\mathrm{cris}(\sigma)}=\frac{x-y}{x+y\left(1-\mathbf{e}^{ut(x-y)}-\mathbf{e}^{vt(x-y)}\right)}. \tag{3.1}$$

For any function $r(t)=\sum_{n=0}^{\infty}c_nt^n$ where $c_n\in\mathbb{C}$, $(r(t)+r(-t))/2$ is equal to the series $\sum_{n=0}^{\infty}c_{2n}t^{2n}$. This simple fact gives that the generating function for D_n is equal to half the sum of (3.1) with (3.1) where v replaced with $-v$. That is, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in D_n}u^{\mathrm{pos}(\sigma)}v^{\mathrm{neg}(\sigma)}x^{\mathrm{cdes}(\sigma)}y^{\mathrm{cris}(\sigma)}\\&=\frac{x-y}{2x+2y\left(1-\mathbf{e}^{ut(x-y)}-\mathbf{e}^{vt(x-y)}\right)}+\frac{x-y}{2x+2y\left(1-\mathbf{e}^{ut(x-y)}-\mathbf{e}^{-vt(x-y)}\right)}.\end{aligned} \tag{3.2}$$

Besides $\mathrm{cdes}(\sigma)$, there are statistics for the hyperoctahedral group B_n motivated by viewing B_n as a Coxeter group [**4, 59, 60, 61**]. Define a linear order Θ on $\{\pm1,\ldots,\pm n\}$ such that

$$1<_\Theta\cdots<_\Theta n<_\Theta -n<_\Theta\cdots<_\Theta -1.$$

for all $n\geq 1$. Define the statistic $\mathrm{des}_B(\sigma)$ on B_n such that

$$\mathrm{des}_B(\sigma)=\chi(n+1<_\Theta\sigma_n)+\sum_i\chi(\sigma_{i+1}<_\Theta\sigma_i).$$

This definition and the linear order Θ naturally arise from interpreting B_n as a Coxeter group [**4, 59**]. A companion to $\mathrm{des}_B(\sigma)$ for $\sigma\in B_n$ is the statistic $\mathrm{ris}_B(\sigma)$ which is defined such that $\mathrm{des}_B(\sigma)+\mathrm{ris}_B(\sigma)=n$. Similar definitions can be made for $\mathrm{inv}_B(\sigma)$ and $\mathrm{coinv}_B(\sigma)$ as well as $\mathrm{comaj}_B(\sigma)$, etc.

To find permutation statistics for $\mathrm{des}_B(\sigma)$ and $\mathrm{ris}_B(\sigma)$ over the hyperoctahedral group and D_n, define a weighting function f_{12} such that $f_{12}(0)=1$ and for $n\geq1$,

$$f_{12}(n)=u^n(-y)(x-y)^{n-1}+v^n(-x)(y-x)^{n-1}.$$

Theorem 3.2.

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in B_n}u^{\mathrm{pos}(\sigma)}v^{\mathrm{neg}(\sigma)}x^{\mathrm{des}_B(\sigma)}y^{\mathrm{ris}_B(\sigma)}=\frac{x-y}{x\mathbf{e}^{tv(y-x)}-y\mathbf{e}^{tu(x-y)}}.$$

Proof. The weighting function f_{12} allows for two types of bricks in a brick tabloid; those weighted with u in every cell and those weighted with v in every cell. The integers in bricks with u will be considered positive while the integers in bricks with v will be considered negative. In any positive brick, the weighting function places a y at the end of the brick and x or $-y$ in every other cell. In any negative brick, the weighting function places an x at the end of the brick and $-x$ or y in every other cell. An example of one $T\in\mathcal{T}_{f_{12}}$ is below.

x	y	y	y	$-x$	$-x$	y	x	y	$-y$	x	y
u	u	u	v	v	v	v	v	u	u	u	u
9	6	8	11	10	7	5	3	1	12	4	2

To every positive brick, apply the brick breaking/combining involution found in the proof of Theorem 3.1 where two bricks are combined if there is a descent and the two bricks are both labeled with u. To every negative brick, scan the cells from left to right looking for either a $-x$ or two consecutive negative bricks where the integer labels between the bricks decrease. Break or combine the bricks accordingly. This process is a sign-reversing weight-preserving involution with fixed points corresponding to elements in B_n where every integer i in a positive brick is interpreted as $(+1, i)$ and every integer i in a negative brick is interpreted as $(-1, i)$. In this way, the u, v, x, and y weights all register the correct statistics. This completes the proof. $\square$

Taking $u = 1$,$v = 0$, and $y = 1$, we find the generating function in Theorem 1.17—this follows because S_n is the subgroup of B_n consisting of those signed permutations with only positive sign. Beck found the special case of the above generating function when $u = v = y = 1$. When taking $v = y = 1$ and $u = -1$, the generating function simplifies to $e^{t(1-x)}$. Taking the coefficient of $t^n/n!$ in this expression gives simply $(1 - x)^n$. This special case was also proved by Beck (she had a less general version at her disposal so noticing this fact was not trivial as it is here). When taking $u = v = 1/2$ and $y = 1$, Reiner found the resulting generating function using upper binomial posets [**61**].

Via the methods which gave (3.2), we have a generating function for D_n:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in D_n} u^{\mathrm{pos}(\sigma)} v^{\mathrm{neg}(\sigma)} x^{\mathrm{des}_B(\sigma)} y^{\mathrm{ris}_B(\sigma)}$$
$$= \frac{x-y}{2x\mathbf{e}^{tv(y-x)} - 2y\mathbf{e}^{tu(x-y)}} + \frac{x-y}{2x\mathbf{e}^{tv(x-y)} - 2y\mathbf{e}^{tu(x-y)}}.$$

Define a homomorphism ξ_8 on the ring of symmetric functions by defining it on e_n such that

$$\xi_8(e_n) = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n}} \frac{(u/v)^{0i_0+\cdots+ki_k}}{[i_0]_{p,q}! \cdots [i_k]_{p,q}!} q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}} [i_0+1]_{w,z} \cdots [i_k+1]_{w,z}.$$

This is the homomorphism found in the proof of Theorem 2.22 with the addition of the factor of the form $[i_0+1]_{w,z} \cdots [i_k+1]_{w,z}$. This factor will allow us to designate some cells as positive while others negative. In the positive cells we will insert a w while in the negative signs we will insert z.

Theorem 3.3.

$$\sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!(y,x;v,u)_{n+1}}$$
$$\times \sum_{\sigma \in B_n} x^{\mathrm{des}_B(\sigma)} y^{\mathrm{ris}_B(\sigma)} u^{\mathrm{maj}_B(\sigma)} v^{\mathrm{comaj}_B(\sigma)} q^{\mathrm{inv}_B(\sigma)} p^{\mathrm{coinv}_B(\sigma)} w^{\mathrm{pos}(\sigma)} z^{\mathrm{neg}(\sigma)}$$
$$= \sum_{k\geq 0} \frac{x^k (w-z)^k}{y^{k+1} \left(w\mathbf{e}_{p,q}^{-tw(u/v)^0} - z\mathbf{e}_{p,q}^{-tz(u/v)^0}\right) \cdots \left(w\mathbf{e}_{p,q}^{-tw(u/v)^k} - z\mathbf{e}_{p,q}^{-tz(u/v)^k}\right)}.$$

Proof. We may form brick tabloids from expanding $[n]_{p,q}!\xi_8(h_n)$ in terms of the elementary symmetric functions in a similar way as found in the proof of Theorem 2.22.

Let us start with a brick tabloid as described in the proof of Theorem 2.22. The powers of w and z appear in the form

$$[i_j+1]_{w,z} = w^{i_j} z^0 + \cdots + w^0 z^{i_j}.$$

This allows us to select one monomial in w and z of degree i_j for each string of consecutive cells in brick labeled with the same power of (u/v). With this choice, place a string of z's in the first cells and w's in the remaining. This filling brick tabloids, together with the filling described in the proof of Theorem 2.22 gives objects whose weighted sum is equal to the application of ξ_8 to $[n]_{p,q}!h_n$. An example of one such brick tabloid may be found below.

-1	-1	-1	-1	1	-1	1	-1	-1	1	1	1
z	w	z	z	w	w	z	w	z	w	z	w
u^0v^0	u^0v^0	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^2v^{-2}	u^3v^{-3}	u^1v^{-1}	u^3v^{-3}	u^3v^{-3}	u^0v^0	u^1v^{-1}
$q^{11}p^0$	q^0p^{10}	q^8p^1	q^7p^1	q^0p^7	q^1p^5	q^0p^5	q^1p^3	q^1p^3	q^0p^2	q^1p^0	q^0p^0
12	1	10	9	2	4	3	6	7	5	11	8

Suppose we have j consecutive cells within a brick with the same power of (u/v) and marked with a z. Instead of writing these integers in decreasing order as prescribed in the proof of Theorem 2.22, let us reverse the order of these j integers so that they are in increasing order. This is done so that cells in a brick marked with the same power of (u/v) are in descending order according to the linear order Θ. For instance, the result of performing this switch to the above object may be found below.

-1	-1	-1	-1	1	-1	1	-1	-1	1	1	1
z	w	z	z	w	w	z	w	z	w	z	w
u^0v^0	u^0v^0	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^2v^{-2}	u^3v^{-3}	u^1v^{-1}	u^3v^{-3}	u^3v^{-3}	u^0v^0	u^1v^{-1}
$q^{11}p^0$	q^0p^{10}	q^8p^1	q^7p^1	q^0p^7	q^1p^5	q^0p^5	q^1p^3	q^1p^3	q^0p^2	q^1p^0	q^0p^0
12	1	9	10	2	4	3	6	7	5	11	8

The same brick breaking/combining sign-reversing weight-preserving involution as in Theorem 2.22 may be now applied to leave a set of fixed points which may be see to correspond to elements in B_n. In fact, all of the proof of Theorem 2.22 follows through in the exact same manner in this case. By employing (1.4), the generating function for

$$\sum_{n\geq 0} \frac{yt^n}{[n]_{p,q}!(y,x;v,u)_{n+1}} \sum_{\sigma\in B_n} x^{\mathrm{des}_B(\sigma)} y^{\mathrm{ris}_B(\sigma)} \times u^{\mathrm{maj}_B(\sigma)} v^{\mathrm{comaj}_B(\sigma)} q^{\mathrm{inv}_B(\sigma)} p^{\mathrm{coinv}_B(\sigma)} w^{\mathrm{pos}(\sigma)} z^{\mathrm{neg}(\sigma)}$$

is equal to

$$\sum_{k\geq 0}(x/y)^k\left(\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\dots,i_k\geq 0\\ i_0+\cdots+i_k=n}}\frac{(u/v)^{0i_0+\cdots+ki_k}}{[i_0]_{p,q}!\cdots[i_k]_{p,q}!}q^{\binom{i_0}{2}+\cdots+\binom{i_k}{2}}\right.$$
$$\left.[i_0+1]_{w,z}\cdots[i_k+1]_{w,z}\right)^{-1}.$$

which in turn may be simplified to look like the statement of the theorem. □

As for D_n, the subgroup consisting of those $\sigma\in B_n$ with $\mathrm{neg}(\sigma)$ an even number, notice that

$$\frac{1}{2}z^{\mathrm{neg}(\sigma)}+\frac{1}{2}(-z)^{\mathrm{neg}(\sigma)}=\begin{cases}z^{\mathrm{neg}(\sigma)} & \text{if } \sigma\in D_n,\\ 0 & \text{if } \sigma\notin D_n.\end{cases}\tag{3.3}$$

Therefore, the generating function in Theorem 3.3 may be specialized for D_n. We have

$$\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!(y,x;v,u)_{n+1}}$$
$$\times\sum_{\sigma\in D_n}x^{\mathrm{des}_B(\sigma)}y^{\mathrm{ris}_B(\sigma)}u^{\mathrm{maj}_B(\sigma)}v^{\mathrm{comaj}_B(\sigma)}q^{\mathrm{inv}_B(\sigma)}p^{\mathrm{coinv}_B(\sigma)}w^{\mathrm{pos}(\sigma)}z^{\mathrm{neg}(\sigma)}$$
$$=\frac{1}{2}\sum_{k\geq 0}\frac{x^k(w-z)^k}{y^{k+1}\left(w\mathbf{e}_{p,q}^{-tw(u/v)^0}-z\mathbf{e}_{p,q}^{-tz(u/v)^0}\right)\cdots\left(w\mathbf{e}_{p,q}^{-tw(u/v)^k}-z\mathbf{e}_{p,q}^{-tz(u/v)^k}\right)}$$
$$+\frac{1}{2}\sum_{k\geq 0}\frac{x^k(w+z)^k}{y^{k+1}\left(w\mathbf{e}_{p,q}^{-tw(u/v)^0}+z\mathbf{e}_{p,q}^{tz(u/v)^0}\right)\cdots\left(w\mathbf{e}_{p,q}^{-tw(u/v)^k}+z\mathbf{e}_{p,q}^{tz(u/v)^k}\right)}.$$

3.2. Words

Much of the work we have done in this monograph so far may be applied to the case of words. A word of length n in the letters $\{a_1,\dots,a_k\}$ is a non-commutative string in $a_1,\dots,a_k$ of length n. This set will be denoted by $\{a_1,\dots,a_k\}^*_n$. The number of words may also be referred to as the number of rearrangements of $a_1,\dots,a_k$. Recent publications have already analyzed the distribution of τ-matches in k-ary words of length n and several nice results has been presented in [**18, 17**] (for the case of τ-avoiding see [**15, 16**]).

Before stating a general result for words with letters in an arbitrary finite set, let us consider elements in $\{1,\dots,k\}^*$ (or any other finite set with a linear order). For this set we may consider statistics depending on descents and rises in the word. For a word $w=w_1\cdots w_n$, let $\mathrm{wdes}(w)$ be the number of times $w_i\geq w_{i+1}$ for $i=1,\dots,n-1$; i.e., the number of weak descents in the word w. The statistic $\mathrm{ris}(w)$ will still count the number of times $w_i<w_{i+1}$ for $i=1,\dots,n-1$ with the convention that w_n always registers a rise in a word of length n.

Let $\xi_9^{f_1}$ be the homomorphism defined on the elementary symmetric functions such that

$$\xi_9^{f_1}(e_n) = (-1)^n \binom{n+k-1}{n} f_1(n) = (-1)^n \binom{n+k-1}{n} (-y)(x-y)^{n-1}.$$

The next theorem will relate this homomorphism to an ordinary generating function for words (by an ordinary generating function, we mean that there is not a factor of the form $n!$ dividing the coefficient of t^n in the generating function).

THEOREM 3.4.

$$\sum_{n=0}^{\infty} t^n \sum_{w \in \{1,\ldots,k\}_n^*} x^{\mathrm{wdes}(w)} y^{\mathrm{ris}(w)} = \frac{x-y}{x - y\,(1 - t(x-y))^{-k}}.$$

PROOF. As usual, we start by applying $\xi_9^{f_1}$ to the homogeneous symmetric functions. We have

$$\begin{aligned}
\xi_9^{f_1}(h_n) &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,(n)}| \xi_9^{f_1}(e_\lambda) \\
&= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} |B_{\lambda,(n)}| \binom{\lambda_1+k-1}{\lambda_1} \cdots \binom{\lambda_\ell+k-1}{\lambda_\ell} f_1(\lambda_1)\cdots f_1(\lambda_\ell).
\end{aligned}$$

To count the above sum combinatorially, start with a brick tabloid of shape (n) and type λ. Basic combinatorial arguments can show that $\binom{n+k-1}{n}$ is the number of words $w = w_1 \cdots w_n$ in $\{1,\ldots,k\}_n^*$ with $w_i \geq w_{i+1}$ for $i = 1,\ldots,n-1$. Thus, we may fill the bricks in the brick tabloid with weakly decreasing sequences of integers from $\{1,\ldots,k\}$ to use the binomial coefficients in the above sum. The product of the weighting functions f_1 combined with the power of $(-1)^{\ell(\lambda)}$ weights the bricks with sequences of x, $-y$, and y such that each brick ends in y and the rest of the cells are labeled with either x or $-y$.

Apply a brick breaking and combining involution where bricks are broken at the first scanned $-y$ or combined when two bricks have a weak decrease between them. Fixed points which are left correspond to words in $\{1,\ldots,k\}_n^*$ with powers of x counting weak descents and powers of y counting rises. A fixed point when $k = 3$ is given below.

x	y	y	x	x	y	x	x	x	x	x	y
3	1	2	3	3	1	3	3	2	2	2	1

We have

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n \sum_{w \in \{1,\ldots,k\}_n^*} x^{\mathrm{wdes}(w)} y^{\mathrm{ris}(w)} &= \xi_9^{f_1}\left(\sum_{n=0}^{\infty} t^n h_n\right) \\
&= \left(\sum_{n=0}^{\infty} t^n \binom{n+k-1}{n} (-y)(x-y)^{n-1}\right)^{-1},
\end{aligned}$$

which, by an application of Newton's binomial formula, is equal to the desired expression. $\square$

The difference between Theorem 3.4 and the results on permutation statistics is the factor of $1/n!$ dividing the coefficient of t^n in the generating functions. This factor of $1/n!$ has been used in previous proofs to allow for subsets of $\{1,\ldots,n\}$ to be associated to each brick. These subsets when read from left to right in a labeled brick tabloid formed a permutation. When dealing with words, we are not interested in relabeling the integers in the cells of a brick to form a permutation and thus this factor is not needed. This indicates why, in general, exponential generating functions (those with this factor of $1/n!$) are used when counting labeled objects and ordinary generating functions (those without this factor of $1/n!$) are used when dealing with unlabeled objects.

Of course there are other choices for weighting functions besides f_1 to use in the definition of the homomorphism $\xi_9^{f_1}$. Many of our other weighting functions may be used. For example, when applying $\xi_9^{f_3}$, one may find that

$$\sum_{n=0}^{\infty} t^n |\{w \in \{1,\ldots,k\}_n^* : w \text{ is even alternating}\}| = (1+t^2)^{k/2} \sec\left(k \arctan(t)\right).$$

where a word $w = w_1 \cdots w_n$ in $\{1,\ldots,k\}_n^*$ is even alternating provided n is even and $w_i < w_{i+1}$ if and only if i is even.

The techniques of appropriately weighting the last brick in a brick tabloid we have shown earlier in the section on the alternating permutations easily prove that

$$\sum_{n=0}^{\infty} t^n |\{w \in \{1,\ldots,k\}_n^* : w \text{ is odd alternating}\}| = \tan\left(k \arctan(t)\right).$$

Thus, these alternating words in $\{1,\ldots,k\}^*$ have analogous generating functions as the alternating permutations of n.

If we wanted to count the number of common weak descents or one rises in a collection of m elements in $\{1,\ldots,k\}_n^*$, this may be done by altering the homomorphism $\xi_9^{f_1}$ to map the n^{th} elementary symmetric to

$$(-1)^n \binom{n+k-1}{n}^m f_1(n) = (-1)^n \binom{n+k-1}{n}^m (-y)(x-y)^{n-1}$$

and then applying this new homomorphism to the homogeneous symmetric functions to find a generating function. The weighting function may be changed here as well to find even more elaborate expressions for m-tuples of words in $\{1,\ldots,k\}_n^*$.

The p,q-analogues we detailed in the last couple of sections do not pass to the case of words as easily. This is because the factor of the form $1/[n]_{q,p}!$ was used to count the inversions which appear between the integers in different bricks in a brick tabloids. In the case of words, we do not have this factor, thus making the job of finding p,q-analogues more difficult.

There are other things which may be done in the case of words, however. Let $a_1,\ldots,a_k$ be non-commuting variables and for $w \in \{a_1,\ldots,a_k\}_n^*$ and $i = 1,\ldots,k$. Define a block of a_i as one or more consecutive a_i's appearing in w. Let $\mathrm{blk}_i(w)$ be the number of blocks of a_i in w and define ξ_{10} be the homomorphism on Λ such that

$$\xi_{10}(e_n) = (-1)^n \sum_{i=1}^{k} (-x_i)(1-x_i)^{n-1}$$

for indeterminates $x_1,\ldots,x_k$. These definitions lead to Theorem 3.5 below.

Theorem 3.5.

$$\sum_{n=0}^{\infty} t^n \sum_{w\in\{a_1,\ldots,a_k\}_n^*} x_1^{\mathrm{blk}_1(w)}\cdots x_k^{\mathrm{blk}_k(w)} = \left(1-\sum_{i=1}^{k}\frac{x_i t}{1-t(1-x_i)}\right)^{-1}.$$

Proof. Applying ξ_{10} to h_n, we have

$$\xi_{10}(h_n)=\sum_{\lambda\vdash n}|B_{\lambda,(n)}|(-1)^{\ell(\lambda)}\prod_{j=1}^{\ell(\lambda)}\sum_{i=1}^{k}(-x_i)(1-x_i)^{\lambda_j-1}.$$

To create brick tabloids in order to deal with the above equation, take a brick tabloid of shape (n) and for each brick choose an integer i between 1 and k. With this choice, place either 1 or $-x_i$ in the nonterminal cells of the brick and $-x_i$ in the terminal cell. Use the powers of -1 in the factor of the form $(-1)^{\ell(\lambda)}$ to reverse the sign on the terminal $-x_i$.

On this collection of decorated brick tabloids we may apply a brick breaking/combining involution where bricks are broken at the first cell containing a $-x_i$ or consecutive bricks are combined if the indeterminates are indexed by the same integer i. A fixed point under this involution may be found below.

1	x_1	1	1	x_3	x_2	x_3	1	1	x_2	1	x_1

A brick of length j ending in x_i in a fixed point can be thought of as j copies of a_i. For example, the above fixed point corresponds to the word $a_1^2a_3^3a_2a_3a_2^3a_1$. Thus, we have that

$$\begin{aligned}\sum_{n=0}^{\infty} t^n \sum_{s\in\{a_1,\ldots,a_k\}_n^*} x_1^{\mathrm{blk}_1(s)}\cdots x_k^{\mathrm{blk}_k(s)} &= \xi_{10}\left(\sum_{n=0}^{\infty}h_n t^n\right)\\ &= \left(\sum_{n=0}^{\infty}t^n\sum_{i=1}^{k}(-x_i)(1-x_i)^{n-1}\right)^{-1},\end{aligned}$$

which is equal to the desired expression. □

Much of the work we have done in Section 2.3 carries through beautifully to the situation of words. For $m\geq 1$, let $\{1,\ldots,m\}_n^*$ be the set of all words of length n in the letters $\{1,\ldots,m\}$. For $v\in\{1,\ldots,m\}_j^*$, we define a word $w\in\{1,\ldots,m\}_j^*$ to have a v-match at place i provided $w_i\cdots w_{i+j-1}$ is equal to v. This is the natural analogue for τ-matches in the case of words. Furthermore, let v-mch(w) be the total number of v-matches in w and v-nlap(w) be the maximum number of nonoverlapping v-matches in w where two v-matches are said to overlap if they contain any of the same letters. We call a word of length n v-saturated if it has the maximum number of v-matches possible for a word of length n.

Just as in the case of permutations, there are two integers dependent on the word v which will be used frequently throughout our study of generating functions for words. For $v\in\{1,\ldots,m\}_j^*$, define k_v such that

$$j+k_v=\min\{n: \text{there exists a } w\in\{1,\ldots,m\}_n^* \text{ with } v\text{-mch}(w)=2\}$$

and let i_v be the least integer larger than 1 such that $i_v k_v \geq j$. Let $f_4(n)$ be equal to

$$\begin{cases} 1 & \text{if } n = 0, \\ m & \text{if } n = 1, \\ (1-x)|\{w \in \{1,\ldots,m\}^*_n : w \text{ is } v\text{-saturated}\}| & \text{if } n = j + \ell i_v k_v \text{ for } \ell \geq 0, \\ -(1-x)|\{w \in \{1,\ldots,m\}^*_n : w \text{ is } v\text{-saturated}\}| & \text{if } n = j + (\ell i_v + 1)k_v \text{ for } \ell \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

and define $\xi_{11}^{f_4}$ as the ring homomorphism on Λ with the property that

$$\xi_{11}^{f_4}(e_n) = (-1)^n f_4(n).$$

THEOREM 3.6. *For $\tau \in S_j$,*

$$\sum_{n=0}^{\infty} t^n \sum_{w \in \{1,\ldots,m\}^*_n} x^{v\text{-nlap}(w)} = \frac{A(t)}{(1-x) + x(1-mt)A(t)}$$

where $A(t) = \sum_{n=0}^{\infty} t^n |\{w \in \{1,\ldots,m\}^*_n : v\text{-mch}(w) = 0\}|$.

PROOF. We have

$$\xi_{11}^{f_4}(h_n) = \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)} B_{\lambda,n} f_4(\lambda_1) \cdots f_4(\lambda_\ell). \tag{3.4}$$

To construct combinatorial objects from (3.4), we may start with a brick tabloid of shape (n) and type λ for some $\lambda \vdash n$ to account for the summand and $B_{\lambda,n}$ term. The factors of the form $(-1)^{\ell(\lambda)}$ and $f_4(\lambda_1)\cdots f_4(\lambda_\ell)$ do a few things to the brick tabloid. First, as usual, the possible lengths of bricks which appear in the brick tabloid are restricted by the function f_4. Then, the powers of x and -1 in these terms place either x or -1 in specific cells. Finally, the factors of the form $|\{w \in \{1,\ldots,m\}^*_n : w \text{ is } v\text{-saturated}\}|$ in the definition of f_4 allow us to arrange the decreasing words associated to each brick to form a v-saturated word. One object formed according to these specifications when $m = 3$ and $v = 1\ 3\ 1$ may be found below.

		-1			x					-1	
2	1	3	1	3	1	1	3	1	3	1	2

The same sign-reversing weight preserving involution as found in Theorem 2.15 may be applied with the caveat that bricks are scanned for an occurrence of the word v instead of the pattern τ. Fixed points under this involution correspond to words with powers of x counting the maximum number of nonoverlapping v-matches. The statement of the theorem follows by simplifing

$$\sum_{n=0}^{\infty} t^n \sum_{w \in \{1,\ldots,m\}^*_n} x^{v\text{-nlap}(w)} = \xi_{11}^{f_4}\left(\sum_{n=0}^{\infty} h_n t^n\right) = \xi_{11}^{f_4}\left(\sum_{n=0}^{\infty} e_n (-t)^n\right)^{-1}$$

with similar steps as given in the proof of Theorem 2.15. $\square$

Before we continue our exposition with an example illustrating the use of Theorem 3.5, we note that implicit in the above proof we have shown that the reciprocal of $A(t) = \sum_{n=0}^{\infty} t^n |\{w \in \{1,\dots,m\}_n^* : v\text{-mch}(w) = 0\}|$ is equal to

$$1 - mt + \sum_{n=0}^{\infty} t^n |\{w \in \{1,\dots,m\}_n^* : w \text{ is } v\text{-saturated}\}| \\ \times \left(\begin{cases} 1 & \text{if } n = j + \ell i_v k_v \text{ for } \ell \geq 0, \\ -1 & \text{if } n = j + (\ell i_v + 1)k_v \text{ for } \ell \geq 0, \\ 0 & \text{otherwise.} \end{cases} \right)$$

which is sometimes useful in computations. As in the case for permutations, the coefficients in the reciprocal of the power series of $A(t)$ are frequently easier to find than directly counting the words with v-mch$(w) = 0$.

For an example of the use of Theorem 3.5, let us count the number of binary sequences—that is, sequences in the letters 0 and 1—according to the maximum nonoverlapping occurrences of the word $v = 0100$. Counting the number of v-saturated permutations is easier than counting the number of words without any occurrences of 0100. Note that $k_v = 3$ and $i_v = 2$ so we only need to count the number of v-saturated words of length $4 + 6\ell$ and $7 + 6\ell$ for $\ell \geq 0$. There is only one v-saturated word appearing in these lengths: $0100100\cdots100$. Therefore, when $v = 0100$,

$$\sum_{n=0}^{\infty} t^n \sum_{w \in \{0,1\}_n^*} x^{v\text{-nlap}(w)} = \frac{1 + t^3}{1 - 2t + t^3 - (1+x)t^4}.$$

The proof of Theorem 3.5 follows through in the same manner when considering more than one word v. That is, for $u, v \in \{1,\dots,m\}_j^*$, let $k_{u,v}$ be the integer such that $k_{u,v} + j$ is equal to

$$\min\{n : \text{there exists a } \sigma \in S_n \text{ with a } u\text{-match appearing before a } v\text{-match}\}.$$

Then if $\mathcal{V}$ is a set of words such that $k_{u,v}$ is constant for all choices of $u, v \in \mathcal{V}$, the statement of Theorem 3.5 holds provided we change $A(t)$ to be equal to $\sum_{n=0}^{\infty} t^n |\{w \in \{1,\dots,m\}_n^* : v\text{-mch}(w) = 0 \text{ for all } v \in \mathcal{V}\}|$. As an example of this, let us consider words in the set $\{1,2,3\}_n^*$ and take $\mathcal{V} = \{3\,1, 3\,2\}$. In this case, $k_{u,v} = 2$ for all $u, v \in \mathcal{V}$. The constant i_v is equal to 2 even though $k_{u,v} \geq j$; it is defined to be larger than 1. The number of permutations of an even length 2ℓ saturated with instances of 3 1 or 3 2 is equal to 2ℓ since for every two consecutive letters we have a choice of either 3 1 or 3 2. Thus, the ordinary generating function refining the number of words in $\{1,2,3\}_n^*$ by the maximum number of nonoverlapping 3 1 or 3 2-matches is equal to

$$\frac{(t^4 - 1)^2}{1 - 3t - 2(x-1)t^2 + (x-3)t^4 + 6t^5 - (x-1)t^6 - (x-2)t^8 - 3t^9}$$

which is a seemingly complicated generating function we found rather easily. Note that since there is no way for the words 3 1 or 3 2 to acutally overlap, the above generating function is really couting a power of x for every occurrence of the word 3 1 or 3 2.

We are able to count the number of common nonoverlapping v-matches in multiple words. We will explain how to find such statistics for two words w and y

which are both elements in $\{1,\ldots,m\}_n^*$ after which it should be apparent how to generalize to more than two words.

The adjustments we must make to Theorem 3.5 are the following. First, let $g_2(n)$ be equal to

$$\begin{cases} 1 & \text{if } n=0, \\ m^2 & \text{if } n=1, \\ (1-x)|\{w \in \{1,\ldots,m\}_n^* : w \text{ is } v\text{-saturated}\}|^2 & \text{if } n = j+\ell i_v k_v, \\ -(1-x)|\{w \in \{1,\ldots,m\}_n^* : w \text{ is } v\text{-saturated}\}|^2 & \text{if } n = j+(\ell i_v+1)k_v, \\ 0 & \text{otherwise.} \end{cases}$$

and define η_2 as the ring homomorphism on Λ with the property that

$$\eta_2(e_n) = (-1)^n g_2(n).$$

Let v-comnlap(w,y) be the maximum number of common nonoverlapping v-matches in the words w and y. Then, in the same vein as many of our previous results, it may be shown that

$$\sum_{n=0}^{\infty} t^n \sum_{w,y \in \{1,\ldots,m\}_n^*} x^{v\text{-comnlap}(w,y)} = \frac{A(t)}{(1-x)+x(1-m^2t)A(t)}$$

where $A(t) = \sum_{n=0}^{\infty} t^n |\{w \in \{1,\ldots,m\}_n^* : v\text{-mch}(w) = 0\}|^2$.

For instance, the ordinary generating function for the number of nonoverlapping $v = 1\ 1\ 0$ matches in two binary sequences w and y is

$$\frac{1+t^3}{1-4t+(2-x)t^3-4t^4}.$$

Since $k_v = 3$ (and thus $i_v = 2$) we are actually have found find the number of times there the two words w and y have the consecutive subword 1 1 0 in the same place.

Much of the work we have done in Section 2.3 carries through easily to the situation of words. For $m \geq 1$, let $\{1,\ldots,m\}_n^*$ be the set of all words of length n in the letters $\{1,\ldots,m\}$. For $v \in \{1,\ldots,m\}_j^*$, we define a word $w \in \{1,\ldots,m\}_j^*$ to have a v-match at place i provided $w_i \cdots w_{i+j-1}$ is equal to v. Similarly if Δ is a set of words in $\{1,\ldots,m\}_j^*$, we define a word $w \in \{1,\ldots,m\}_j^*$ to have a Δ-match at place i provided $w_i \cdots w_{i+j-1} \in \Delta$. In the case of words, these are the natural analogues of τ-matches and Υ-matches for permutations that we defined in section 2.

Furthermore, let v-mch(w) (Δ-$mch(w)$) be the total number of v-matches (Δ-matches) in w and v-nlap(w) (Δ-$nlap(w)$) be the maximum number of nonoverlapping v-matches (Δ-matches) in w where two v-matches (Δ-matches) are said to overlap if they contain any of the same letters.

Now suppose that Δ is a set of words in $\{1,\ldots,m\}_j^*$. The by analogy with our definitions in Section 2 for subsets Υ of S_j, we define the following. If $v = v_1 \cdots v_n \in \{1,\ldots,m\}^*$, we define

$$\text{Mch}_\Delta(v) = \{i : \text{red}(v_{i+1}\cdots v_{i+j}) \in \Delta\} \qquad \text{and}$$
$$I_\Delta = \{1 \leq i < j : \text{there exists a word } v \text{ such that } \text{Mch}_\Delta(v) = \{0,i\}\}.$$

Let I_Δ^* be the set of all words with letters in the set I_Δ. We let ϵ denote the empty word. If $w = w_1 \cdots w_n \in I_\Delta^*$ is word with n-letters, we define

$$\ell(w) = n, \qquad \sum w = \sum_{i=1}^{n} w_i, \qquad \text{and} \qquad ||w|| = j + \sum w.$$

In the special case where $w = \epsilon$, we let $\ell(w) = 0$ and $\sum w = 0$. Let

$$A_\Delta = \{w \in I_\Delta^* : \ell(w) \geq 2 \text{ and } \sum w < j\} \qquad \text{and}$$

$$B_{u,\Delta} = \{w_1 \cdots w_n \in I_\Delta^* : \sum w_2 \cdots w_n + \sum u < j \leq \sum w_1 \cdots w_n + \sum u\}$$

for each word $u \in I_\Delta^*$ with $\sum u < j$.

Form a new alphabet

$$K_\Delta = \{\overline{u} : u \in I_\Delta\} \cup \{\overline{w} : w \in A_\Delta\}.$$

We let $\Psi : K_\Delta^* \to I_\Delta^*$ be the function such that $\Psi(\epsilon) = \epsilon$ and $\Psi(\overline{w_1} \cdots \overline{w_n}) = w_1 \ldots w_n$. Define $\overline{J}_\Delta$ in the following manner.

(1) $\epsilon \in \overline{J}_\Delta$.
(2) $\overline{v} \in \overline{J}_\Delta$ for all $v \in I_\Delta$.
(3) If $\overline{w_1} \cdots \overline{w_n} \in \overline{J}_\Delta$, then $\overline{u}\ \overline{w_1} \cdots \overline{w_n} \in \overline{J}_\Delta$ for all $u \in B_{w_1,\Delta}$.
(4) The only words in $\overline{J}_\Delta$ are the result of applying one of the above rules.

Take $J_\Delta = \Psi(\overline{J}_\Delta)$.

The final set we would like to define is as follows. Let

$$\mathcal{P}_w^\Delta = \{\sigma \in S_{||w||} : \mathrm{Mch}_\Delta(\sigma) = \{0, w_1, w_1 + w_2, \ldots, w_1 + w_2 + \cdots + w_n\}\}$$

and $\mathcal{P}_\epsilon^\Delta = \{\Delta\}$.

For example, if $\Delta = \{3\ 1\ 2, 2\ 1\ 3, 1\ 3\ 2\}$ and $m = 5$, then has

$$\mathrm{Mch}_\Delta(3\ 1\ 2\ 1\ 3\ 2\ 5\ 1\ 3\ 2\ 4) = \{0, 2, 3, 7\}$$

so that 3 1 2 1 3 2 5 1 3 2 4 is an element of $\mathcal{P}_{2\ 1\ 4}^\Delta$.

Next we define a homomorphism on the ring of symmetric functions by defining it on e_n for $n \geq 0$. Let $V(0) = 1$, $V(1) = m$, and

$$V(n) = (1 - x) \sum_{\omega \in J_\Delta, ||\omega|| = n} (-1)^{\overline{\ell}(\omega)} |\mathcal{P}_\omega^\Delta|$$

otherwise. Let ξ_Δ be the ring homomorphism on Λ with the property that

$$\xi_\Delta(e_n) = (-1)^n V(n).$$

With these definitions, we have the following theorem.

Theorem 3.7. *For any set of words $\Delta \subseteq \{1, \ldots, m\}_j^*$ where $j > 1$,*

$$A^\Delta(t) = 1 + \sum_{n=1}^{\infty} t^n |\{v \in \{1, \ldots, m\}_n^* : Mch_\Delta(v) = \emptyset\}|$$
$$= \frac{1}{1 - mt + \sum_{n=2}^{\infty} t^n \sum_{w \in J_\Delta, ||w|| = n} (-1)^{\overline{\ell}(w)} |\mathcal{P}_w^\Delta|},$$

and

$$\sum_{n=0}^{\infty} t^n \sum_{v \in \{1, \ldots, m\}_n^*} x^{\Delta\text{-}nlap(v)} = \frac{A^\Delta(t)}{(1 - x) + x(1 - mt) A^\Delta(t)}.$$

We note that in the special case where Δ consists of a single word v, Kitaev and Mansour [**50**] proved

$$\sum_{n=0}^{\infty} t^n \sum_{v\in\{1,\dots,m\}^*_n} x^{v\text{-}nlap(v)} = \frac{A^{\Delta}(t)}{(1-x)+x(1-mt)A^{\Delta}(t)}.$$

PROOF. We begin as in the proof of Theorem 2.15 by applying the homomorphism ξ^{Δ} to h_n and describing the result in terms of combinatorial objects. We have

$$\begin{aligned}
\xi^{\Delta}(h_n) &= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\xi^{\Delta}(e_\lambda)\\
&= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\prod_{i=1}^{\ell(\lambda)}(-1)^{\lambda_i}V(\lambda_i)\\
&= \sum_{\lambda\vdash n}(-1)^{\ell(\lambda)}B_{\lambda,n}V(\lambda_1)\cdots V(\lambda_\ell). \qquad (3.5)
\end{aligned}$$

From (3.5) we will build a set $\mathcal{T}_\Delta$ of decorated weighted brick tabloids as follows.

The summand in (3.5) selects a partition λ of n. Use the $B_{\lambda,n}$ term to choose a brick tabloid T of shape (n) filled with bricks $b_1,\dots,b_\ell$, reading from left to right, that induce the partition λ. At this point, we are left with factors of the form $(-1)^{\ell(\lambda)}$ and $V(\lambda_1)\cdots V(\lambda_\ell)$ to aid in the construction of elements in the set $\mathcal{T}_\Delta$.

In dealing with a brick b of length 1, we can choose an integer between 1 and m coming from the term $V(|b|)=m$ and we assign a power of -1 coming from the factor of the form $(-1)^{\ell(\lambda)}$. The factor of $(-1)^{|b|}$ which appears in $\xi^{\Delta}(\lambda_1)\cdots\xi^{\Delta}(\lambda_\ell)$ provides an additional power of -1. These two powers of -1 cancel each other out. Thus, when using the terms in (3.5) to fill the cell in a brick of length 1, we simply place the assigned integer between 1 and m and nothing else.

For each brick b_i with $|b_i|>1$, we have a power of -1 coming from the term $(-1)^{\ell(\lambda)}$ and one $\xi^{\Delta}(|b_i|)$ term. We use the $\xi^{\Delta}(|b_i|)$ term to do the following things. First, we pick at $w\in J_\Delta$ such that $||w||=|b_i|$ and a word $u\in\mathcal{P}^{\Delta}_w$ to place in the cells of b_i. If $w=i_1i_2\cdots i_k$, then let $\overline{u}=\overline{u}_1\cdots\overline{u}_t$ be the word in $\overline{J}_\Delta$ such that $\Psi(\overline{u})=w$ and let $u_i=\Psi(\overline{u}_i)$ and $j_i=\sum u_i$ for $i=1,\dots,t$. Then we place a -1 on top of the cells $j_1, j_1+j_2,\cdots, j_1+j_2+\cdots j_t$ in b_i. This accounts for the term $(-1)^{\overline{\ell}(w)}$. Finally, the product of the -1 coming from the $(-1)^{\ell(\lambda)}$ and the term $(1-x)$ coming from $\xi^{\Delta}(|b_i|)$, leaves us with a term $x-1$. Thus we can either put an x or a -1 on the last cell. Note that our definitions ensure that we can recover w from u so that the sets $\mathcal{P}^{\Delta}_w$ such that $w\in J_\Delta$ are pairwise disjoint.

To re-cap, our construction gives a that elements $T\in\mathcal{T}_\Delta$ are brick tabloids filled such that

- each cell of T is filled with a letter from $\{1,\dots,m\}$,
- a brick of length 1 contains one integer,
- a brick b of length $m\geq 2$ contains a word which is a u-match for some $u\in\mathcal{P}^{\Delta}_w$ where $w=i_1i_2\cdots i_k\in J_\Delta$ and $||w||=m$. Then we find $\overline{u}=\overline{u}_1\cdots\overline{u}_t\in\overline{J}_\Delta$ such that $\Psi(\overline{u})=w$, and set $u_i=\Psi(\overline{u}_i)$ and $j_i=\sum u_i$ for $i=1,\dots,t$. Then we put -1 on top of the cells $j_1, j_1+j_2,\dots, j_1+j_2+\cdots j_t$ in b and a choice of either x or -1 for the terminal cell of b.
- if there is no $w\in J_\Delta$ such that $||w||=m$, then there are no bricks of length m.

One object formed according to these specifications when $m = 3$ and $\Delta = \{1\ 3\ 1\}$ may be found below.

		-1			x					-1	
2	1	3	1	3	1	1	3	1	3	1	2

The same sign-reversing weight preserving involution as found in 2.18 may be applied with the caveat that bricks are scanned for an occurrence of a word $v \in \Delta$ instead of the pattern $\Delta \in \Upsilon$. Fixed points under this involution correspond to words with powers of x counting the maximum number of nonoverlapping Δ-matches.

We note that in the special case where $I_\Delta = \emptyset$, then the only bricks allowed are of size 1 and size j. That is, since $J_\Delta = \{\epsilon\}$ in this case, we will have $V(n) = 0$ for $n \geq 2$ and $n \neq j$ and

$$V(j) = (1-x)(-1)^{\overline{\ell}(\epsilon)}|\mathcal{P}^{\Delta}_{\epsilon}| = (1-x)|\Delta|. \tag{3.6}$$

In this case, the only cases of the involution that can be applied are Cases A and B.

The statement of the theorem follows by simplifying

$$\sum_{n=0}^{\infty} t^n \sum_{w \in \{1,\ldots,m\}^*_n} x^{\Delta\text{-}nlap(w)} = \xi^\Delta \left(\sum_{n=0}^{\infty} h_n t^n \right) = \xi^\Delta \left(\sum_{n=0}^{\infty} e_n (-t)^n \right)^{-1}$$

with similar steps as given in the proof of Theorem 2.15. □

Before we continue our exposition with an example illustrating the use of 3.7. Suppose that $\Delta = \{v\}$ and we simply write $\mathcal{P}^v_\omega$ for $\mathcal{P}^\Delta_\omega$. Then we have shown that $A^v(t)$ is equal to

$$\sum_{n=0}^{\infty} t^n |\{w \in \{1, \ldots, m\}^*_n : v\text{-mch}(w) = 0\}| = \left(1 - mt + \sum_{\omega} (-1)^{\ell(\omega)} |\mathcal{P}^v_\omega| \, t^{\|\omega\|} \right)^{-1}$$

which is sometimes useful in computations. As in the case for permutations, the coefficients in the reciprocal of the power series of $A^v(t)$ are potentially easier to find than directly counting the words with v-$mch(w) = 0$.

For an example of the use of Theorem 3.7, let us count the number of binary sequences—that is, sequences in the letters 0 and 1—according to the maximum nonoverlapping occurrences of the word $v = 0100$. Counting the number of elements in $\mathcal{P}^v_\omega$ is easier than counting the number of words without any occurrences of 0100. Note that $I_v = \{3\}$. There words we are counting can only take one form in this case; that is, we are considering the words of the form $0100100 \cdots 100$. Therefore, when $v = 0100$,

$$\sum_{n=0}^{\infty} t^n \sum_{w \in \{0,1\}^*_n} x^{v\text{-nlap}(w)} = \frac{1 + t^3}{1 - 2t + t^3 - (1+x)t^4}.$$

The proof of 3.7 follows through in the same manner when considering Δ-matches for k-tuples of words rather than for a single word. That is, assume Δ is a set of words in $\{1, \ldots, m\}^*_j$. The predictable definitions of the sets I^k_Δ, A^k_Δ, etc., may be made. Then, in order to find the exponential generating function for the number of permutations in $(\{1, \ldots, m\}^*_n)^k$ refined by the maximum number of nonoverlapping common Δ-matches, we define a homomorphism on the ring of symmetric functions by defining it on e_n for $n \geq 0$. Let $V_k(0) = 1$, $V_k(1) = m^k$,

$$V_k(n) = (1-x) \sum_{\omega \in J^k_\Delta, ||\omega|| = n} (-1)^{\overline{\ell}(\omega)} |\mathcal{P}^{\Delta,k}_\omega|$$

otherwise. Let $\xi_{\Delta,k}$ be the ring homomorphism on Λ with the property that

$$\xi_{\Delta,k}(e_n) = (-1)^n V_k(n).$$

With these definitions, we can use essentially the same proof as in 2.20 to prove that

$$\xi_{\Delta,k}(h_n) = \sum_{(u^1,\ldots,u^m) \in (\{1,\ldots,m\}^*)^k} x^{\Delta\text{-}comnlap(u^1,\ldots,u^k)}.$$

Thus we have the following.

Theorem 3.8. *For any set of permutations $\Upsilon \subseteq S_j$ where $j > 1$,*

$$\begin{aligned} A^{\Delta,k}(t) &= \sum_{n=0}^{\infty} t^n |\{(u^1,\ldots,u^k) \in (\{1,\ldots,m\}^*)^k : \Delta\text{-}commch(u^1,\ldots,u^k) = 0\}| \\ &= \frac{1}{1 - m^k t + \sum_{n=2}^\infty t^n \sum_{w \in J_{\Delta,k}, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}^{\Delta,k}_w|}, \end{aligned}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} t^n \sum_{(u^1,\ldots,u^k) \in (\{1,\ldots,m\}^*)^k} x^{\Delta\text{-}comnlap(u^1,\ldots,u^k)} \\ &= \frac{1}{1 - m^k t + (1-x)\sum_{n=2}^\infty t^n \sum_{w \in J_{\Delta,k}, ||w||=n} (-1)^{\overline{\ell}(w)} |\mathcal{P}^{\Delta,k}_w|} \\ &= \frac{A^{\Delta,k}(t)}{(1-x) + x(1-m^k t) A^{\Delta,k}(t)}. \end{aligned} \tag{3.7}$$

For example, suppose that $m = 2$, $k = 2$, and $\Delta = \{v\}$ where $v = 2\ 1\ 2$. It is easy to see that $I^2_\Delta = \{2\}$ and $J^2_\Delta = \{2\}^*$. Moreover it is not difficult to see that $|\mathcal{P}^{\Delta,2}_{2^n}| = 1$ for all n. Thus

$$\begin{aligned} A^{\Delta,2}(t) &= \sum_{n=0}^{\infty} t^n |\{(u^1,u^2) \in (\{1,2\}^*)^2 : \Delta\text{-}commch(u^1,u^2) = 0\}| \\ &= \frac{1}{1 - 4t + \sum_{n=0}^\infty t^{3+2n}(-1)^n} \\ &= \frac{1+t^2}{1-4t+t^2-3t^3}. \end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} t^n \sum_{(u^1,u^2) \in (\{1,2\}^*)^2} x^{\Delta\text{-}comnlap(u^1,u^2)} = \frac{1+t^2}{1-4t+t^2-3t^3-xt^3}.$$

3.3. Fibonacci numbers

Generating functions for certain linear recurrence equations with constant coefficients can be found from our techniques. There are simpler ways of finding such generating functions; however, it is curious that the same machinery which produced all of our previous results may be applied in this way.

As a nice demonstration, we will find a generating function for the Fibonacci numbers. They are are defined by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with the initial conditions $F_1 = F_2 = 1$. The homomorphism which will give the generating function for the Fibonacci numbers is ξ_{12} defined such that $\xi_{12}(e_0) = \xi_{12}(e_1) = 1$, $\xi_{12}(e_2) = -1$, and $\xi_{12}(e_n) = 0$ if $n \geq 3$.

THEOREM 3.9.

$$\sum_{n=0}^{\infty} F_n t^n = \frac{t}{1-t-t^2}.$$

PROOF. By using the definition of ξ_{12} on e_n, we have

$$\xi_{12}(h_n) = \sum_{\substack{\lambda \vdash n \text{ has} \\ \text{parts either 1 or 2}}} |B_{\lambda,(n)}|.$$

Thus, the application of ξ_{12} on h_n simply counts the number of brick tabloids of shape (n) where bricks must either have length 1 or 2.

On the other hand, the Fibonacci number F_{n+1} is also equal to the number of brick tabloids of shape (n) with bricks have length either 1 or 2. This may be seen inductively; the last brick may have length 1 in which case the remaining cells give F_n, or the last brick may have length 2 in which case the remaining cells give F_{n-1}. In addition, these objects satisfy the necessary initial conditions. Therefore, $\xi_{12}(h_n) = F_{n+1}$. Employing (1.4),

$$\sum_{n=0}^{\infty} F_n t^n = t \sum_{n=0}^{\infty} F_{n+1} t^n = t \left(\sum_{n=0}^{\infty} \xi_{12}(e_n)(-t)^n \right)^{-1} = \frac{t}{1-t-t^2}. \qquad \square$$

There are other linear recurrence equations with constant coefficients which can be found in this manner. The same ideas used in Theorem 3.9 apply to any linear recurrence equation which can be expressed combinatorially in terms of brick tabloids like the Fibonacci numbers. Certain weights may be assigned to each brick by appropriately modifying the homomorphism defined on the elementary symmetric functions. Thus, all of the linear recurrence equations for which generating functions can be found in the same way as above are of the form

$$F_n = a_1 F_{n-1} + \cdots + a_k F_{n-k}$$

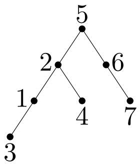
for constants a_k provided the appropriate initial conditions are given to match the combinatorial description of the numbers F_n.

3.4. The exponential formula

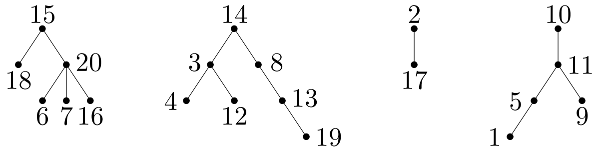
Often times in combinatorics one deals with objects which are constructed from smaller components. For example, a set partition of $\{1, \ldots, n\}$ may be constructed from smaller sets. The number of graphs on n nodes are built from smaller, connected components. The permutations of n with cycles of length $1, 2$, or 3 are built from smaller cycles of length $1, 2$, and 3. The exponential formula gives generating functions for these situations.

For a positive integer n, let $\mathcal{P}_n$ be a set of "pictures" which have, among other things, n circles labeled with $\{1, \ldots, n\}$. By a picture, we simply mean that there is some sort of structure associated with the n labeled circles. Let $\mathcal{L}_n$ be the set of labeled objects with n total circles created with smaller components found in one of $\mathcal{P}_1, \ldots, \mathcal{P}_n$ where the numbers on the circles found in the pictures have been relabeled to give a set partition of $\{1, \ldots, n\}$. For $s \in \mathcal{L}_n$, define the statistic $\mathrm{pic}(s)$ to count the number of components used to build s.

As an example, suppose each element in $\mathcal{P}_n$ is a labeled tree with nodes the n labeled circles. Below we display an element of $\mathcal{P}_7$ in this case.



Objects in the set $\mathcal{L}_n$ are built from components found in $\mathcal{P}_1, \ldots, \mathcal{P}_n$; in this example, labeled forests on n nodes. An element in $\mathcal{L}_{20}$ is shown below where the trees are written in decreasing order according to their smallest element.



If s is the object depicted above, then $\mathrm{pic}(s) = 4$ since it is built from 4 pictures of trees.

A homomorphism can give a relationship between the generating function for pictures and generating function for composite objects. Define ξ_{13} on the elementary symmetric function e_n such that

$$\xi_{13}(e_n) = \frac{(-1)^n}{n!} \sum_{m=1}^{n} \sum_{\substack{i_1, \ldots, i_m \geq 1 \\ i_1 + \cdots + i_m = n}} (-x)^m \binom{n}{i_1, \ldots, i_m} \frac{|\mathcal{P}_{i_1}| \cdots |\mathcal{P}_{i_m}|}{m!}.$$

THEOREM 3.10 (The exponential formula).

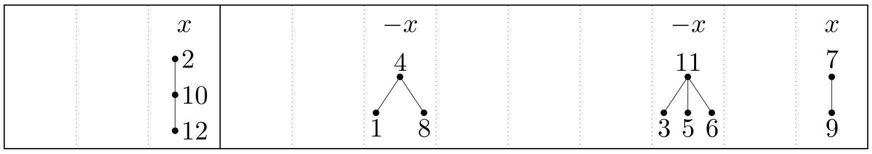
$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{s \in \mathcal{L}_n} x^{\mathrm{pic}(s)} = \mathbf{e}^{x\mathcal{P}(t)}$$

where $\mathcal{P}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} |\mathcal{P}_n|$ is the generating function for the number of pictures with n circles.

PROOF. The homomorphism ξ_{13} tells exactly how to weight a brick of length n when applying ξ_{13} to $n!h_n$. The factor of $(-1)^n$ assigns -1 to every brick, the $1/n!$ term assigns a subset of the first n integers to each brick. We will now describe how rest of the factors in the definition of ξ_{13} may be used to fill bricks. A picture will be placed in some of the cells in a brick of length n while others will remain empty. Before we begin inserting pictures into cells, we will dictate which integers are to label the circles on each picture.

Select an integer m between 1 and n and positive integer indices $i_1, \ldots, i_m$ which sum to n. Use the factor of $|\mathcal{P}_{i_1}| \cdots |\mathcal{P}_{i_m}|$ in the definition of ξ_{13} to select pictures $p_{i_1}, \ldots, p_{i_m}$ with $i_1, \ldots, i_m$ circles. We are given a subset of the first positive integers of size n, say $\{j_1, \ldots, j_n\}$. Use the binomial coefficient $\binom{n}{i_1,\ldots,i_m}$ to replace the numbers $1, \ldots, i_\ell$ on picture p_{i_ℓ} with elements of $\{j_1, \ldots, j_n\}$ such that each picture has distinct integer labels, the union of which is equal to $\{j_1, \ldots, j_n\}$.

Use the $\frac{1}{m!}$ term in the definition of ξ_{13} to sort these pictures containing the re-indexed labels by their smallest element. Suppose that when placed in increasing order according to the smallest element, the pictures are $p_{k_1}, \ldots, p_{k_m}$. Place p_{k_1} in cell number k_1 of the brick when reading left to right. Place p_{k_2} in cell number $k_1 + k_2$ reading left to right. In general, place p_{k_ℓ} in cell number $k_1 + \cdots + k_\ell$ reading left to right. In each one of the cells which now contains a picture, place one factor of $-x$. Since there must be a picture in the last cell in any brick, so must there be a $-x$. Use the factor -1 given to each brick to change the sign on this terminal $-x$ to x. At this point we have accounted for everything in the definition of ξ_{13} to decorate these brick tabloids. An example of one construction when taking pictures to be trees is found below.



The weight of the object comes from the powers of $-x$ or x it contains.

Apply the following sign-reversing weight-preserving involution. Scan the bricks from left to right looking for a $-x$ or two consecutive bricks where the last picture in the first brick contains an larger integer than the first picture in the second brick. If a $-x$ is scanned first, break the brick into two immediately after the $-x$ and reverse the sign on the $-x$. If two consecutive bricks with the appropriate order of pictures is scanned first, then it must be the case that the first brick of the two contains only one picture—otherwise it would contain a $-x$. In this situation, combine the two consecutive bricks and reverse the sign on the x now appearing in the middle of a brick.

This involution is easily seen to be sign-reversing and weight-preserving. The fixed points cannot have any $-x$ labels and hence all bricks must contain one and only one picture. Furthermore, in a fixed point, the pictures when read from left to right must be written in increasing order according to their largest elements. These fixed points correspond to objects in $\mathcal{L}_n$. By (1.4), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{s \in \mathcal{L}_n} x^{\mathrm{pic}(s)} \\
&= \left(1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{m=1}^{n} \sum_{i_1+\cdots+i_m=n} (-x)^m \binom{n}{i_1, \ldots, i_m} \frac{|\mathcal{P}_{i_1}| \cdots |\mathcal{P}_{i_m}|}{m!}\right)^{-1} \\
&= \left(\sum_{m=0}^{\infty} \frac{(-x)^m}{m!} \sum_{n=m}^{\infty} \sum_{i_1+\cdots+i_m=n} t^{i_1+\cdots+i_m} \frac{|\mathcal{P}_{i_1}| \cdots |\mathcal{P}_{i_m}|}{i_1! \cdots i_m!}\right)^{-1}
\end{aligned}$$

which may be seen to equal $\mathbf{e}^{x\mathcal{P}(t)}$. □

Now we will exhibit a few of the standard uses of Theorem 3.10. As a first application we will enumerate set partitions. A set partition of n is a collection of pairwise disjoint nonempty sets $B_1, \ldots, B_k$ with union equal to $\{1, \ldots, n\}$. Below we show a set partition of 12 with sets written in decreasing order according to their smallest element:

$$\{9\}, \{4, 12\}, \{2, 5, 7, 10, 11\}, \{1, 3, 6, 8\}.$$

If s is a set partition of n into k sets, then we say that s has k parts. The number of set partitions of n with k parts is usually denoted $S(n, k)$ and referred to as the Stirling number of the second kind. The number of set partitions of n with any number of parts is called a Bell number.

To count the number of set partitions by parts, take $\mathcal{P}_n$ to contain only one picture, the picture with n circles labeled with $1, \ldots, n$ where the circles are written in a row. It follows that elements in $\mathcal{L}_n$ built with components in $\mathcal{P}_1, \ldots, \mathcal{P}_n$ with n total circles is equal to the set of all set partitions of n. In this case, we have $\mathcal{P}(t)$, the (exponential) generating function starting at $n = 1$ for the number of pictures in $\mathcal{P}_n$, is equal to $\mathbf{e}^t - 1$. Theorem 3.10 shows that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=1}^{n} S(n, k) x^k = \mathbf{e}^{x(\mathbf{e}^t - 1)}. \tag{3.8}$$

In particular, taking the coefficient of x^k in the expansion of the above expression, it follows that

$$\sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!} = \frac{(\mathbf{e}^t - 1)^k}{k!}.$$

Furthermore, by taking $x = 1$ in (3.8), one may find the (exponential) generating function for the Bell numbers.

Let us now find a generating function registering the cycle structure of permutations in the symmetric group. Any permutation in S_n when written in cyclic notation may be considered a composite object built from cycles. All we mean by this, for example, is that the permutation

$$(12\ 8)(4\ 9\ 6\ 3)(5\ 10\ 11\ 2)(7\ 1),$$

written such that cycles decrease according to their smallest element, may be considered a composite object built from two cycles of length 2 and two cycles of length 4.

Let $\mathrm{cyc}_i(\sigma)$ be the number of cycles of length i in $\sigma \in S_n$. For the permutation σ displayed above, $\mathrm{cyc}_i(\sigma) = 2$ if i is 2 or 4 and $\mathrm{cyc}_i(\sigma) = 0$ otherwise. Let us take pictures in $\mathcal{P}_n$ to consist of a cycle together with one power of q_n for some indeterminates $q_1, q_2, \ldots$. These indeterminates q_n may be thought of as an extra weight on each picture. Although we did not state this in Theorem 3.10 for ease of exposition, we have the ability to assign such indeterminates to each picture in $\mathcal{P}_n$ if we so desire; the involution which proved Theorem 3.10 does not affect any of the pictures in a brick tabloid and hence is still weight-preserving with respect to those indeterminates. By Lemma 1.4, there are $(n-1)!$ cycles of length n and therefore $|\mathcal{P}_n| = q_n(n-1)!$. Objects in $\mathcal{L}_n$ are permutations in S_n written in cyclic notation with powers of q_i counting the number of cycles of length i. Therefore, in this situation,

$$\mathcal{P}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} |\mathcal{P}_n| = q_1 t + q_2 \frac{t^2}{2} + q_3 \frac{t^3}{3} + \cdots$$

Thus, taking $x = 1$ in Theorem 3.10 (the power of x in Theorem 3.10 would count the number of cycles in a permutation and we are already counting this with the indeterminates q_i) gives

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} q_1^{\mathrm{cyc}_1(\sigma)} q_2^{\mathrm{cyc}_2(\sigma)} \cdots = \mathbf{e}^{q_1 t + q_2 \frac{t^2}{2} + q_3 \frac{t^3}{3} + \cdots}. \tag{3.9}$$

This is known as the generating function for the cycle index polynomial. By taking special values of $q_1, q_2, \ldots$, this expression can produce some interesting generating functions. For example, if we let $\mathrm{cyc}(\sigma)$ be the total number of cycles in σ and let $q_1 = q_2 = \cdots = x$ in (3.9), we find that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{cyc}(\sigma)} = \mathbf{e}^{x(t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots)} = \mathbf{e}^{x \ln((1-t)^{-1})} = (1-t)^{-x}.$$

An expansion of the right hand side of the above expression by Newton's binomial theorem shows that $\sum_{\sigma \in S_n} x^{\mathrm{cyc}(\sigma)} = (x+0)(x+1)\cdots(x+(n-1))$.

By taking certain values of q_n in (3.9) to be equal to zero, we are restricting the appearances of certain cycles. Let j be a positive integer. It is not difficult to show that the permutations $\sigma \in S_n$ with the property that $\sigma^j = 1$ are those permutations with cycles of length dividing j (here, σ^j means σ is composed with itself j times and 1 is the identity permutation). To find a generating function for those permutations in S_n with $\sigma^j = 1$, take $q_n = x$ if $j \mid n$ and $q_n = 0$ otherwise to find that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\substack{\sigma \in S_n \\ \sigma^j = 1}} x^{\mathrm{cyc}(\sigma)} = \prod_{n \mid j} \mathbf{e}^{x t^n / n}.$$

For example, in the special case of $j = 2$, the generating function registering a power of x for each cycle in an involution in S_n is $\mathbf{e}^{x(t + t^2/2)}$.

Let $\mathcal{G}_n$ be the set of undirected graphs on n labeled nodes with no loops or multiple edges. Since between any two nodes there is a choice of either placing an edge or not, there are $2^{\binom{n}{2}}$ total elements in $\mathcal{G}_n$. If we let $\mathcal{P}(t)$ be the generating function for the number of connected elements in $\mathcal{G}_n$, taking $x = 1$ in Theorem 3.10 shows that

$$\sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{t^n}{n!} = \mathbf{e}^{\mathcal{P}(t)}.$$

Thus, we have found the generating function $\mathcal{P}(t)$. Using this, we can refine elements in $\mathcal{G}_n$ by their number of components with a second application of Theorem 3.10. We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\substack{g \in \mathcal{G}_n \text{ has } k \\ \text{connected components}}} x^k = \mathbf{e}^{x \ln\left(\sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{t^n}{n!}\right)} = \left(\sum_{n=0}^{\infty} 2^{\binom{n}{2}} \frac{t^n}{n!}\right)^x. \tag{3.10}$$

There is an unlabeled version of the exponential formula which also fits into our framework. Once again take $\mathcal{P}_n$ to be a set of pictures which have, among other things, n circles. This time however, the circles are not to be labeled with integers. Arbitrarily give P_n some linear order. Let $\mathcal{U}_n$ be the set of unlabeled objects with n total circles built from components found in one of $\mathcal{P}_1, \ldots, \mathcal{P}_n$. For instance, in the case where P_n contains unlabeled trees on n nodes, $\mathcal{U}_n$ contains the unlabeled forests on n nodes. Just as in the case of labeled objects, for $s \in \mathcal{U}_n$, $\text{pic}(s)$ will be the number of pictures used to create s.

Let ξ_{14} be the homomorphism defined on the elementary symmetric function such that

$$\xi_{14}(e_n) = (-1)^n \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + 2i_2 + \cdots + ni_n = n}} \binom{|\mathcal{P}_1|}{i_1} \cdots \binom{|\mathcal{P}_n|}{i_n} (-x)^{i_1 + \cdots + i_n}$$

where the binomial coefficient $\binom{|\mathcal{P}_j|}{i_j}$ is equal to 0 if $i_j > |\mathcal{P}_j|$.

Theorem 3.11.

$$\sum_{n=0}^{\infty} t^n \sum_{s \in \mathcal{U}_n} x^{\text{pic}(s)} = \prod_{i \geq 1} \frac{1}{(1 - xt^i)^{|\mathcal{P}_i|}}.$$

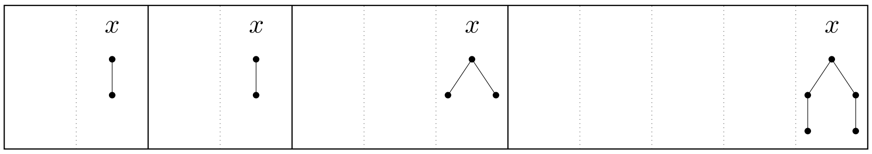
Proof. Expanding h_n in terms of the elementary symmetric functions, we have

$$\xi_{14}(h_n) = \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} |B_{\lambda,(n)}| \xi_{14}(e_{\lambda_1}) \cdots \xi_{14}(e_{\lambda_\ell}). \tag{3.11}$$

Use the sum and the $|B_{\lambda,(n)}|$ term in above equation to select a brick tabloid of shape (n), say T. The factor of $(-1)^n$ in the definition of ξ_{14} and the factor of $(-1)^{n-\ell(\lambda)}$ in (3.11) combine to give a power of (-1) for every brick in T. The summand in the definition of ξ will be used to fill some cells in each brick with a picture and $-x$ just as in the proof of Theorem 3.10.

First, for a brick of size k, choose $i_1, \ldots, i_k$ nonnegative integers such that $i_1 + 2i_2 + \cdots + ki_k = k$. This uses the summand in the definition of ξ_{14}. For $j = 1, \ldots, k$, select i_j different pictures in $\mathcal{P}_{i_j}$ to be placed in the brick, exhausting the multinomial coefficients in the definition of ξ_{14}. Sort these pictures in decreasing order first according to the number of circles, then according to the arbitrary linear order imposed on $\mathcal{P}_n$. Suppose that when this is done, the pictures are $p_{k_1}, \ldots, p_{k_m}$. Place p_{k_j} in cell number $k_1 + \cdots + k_j$ reading left to right. In each one of the cells which now contains a picture, place one factor of $-x$. Since there must be a picture in the last cell in any brick, so must there be a $-x$. Use the factor -1 given to each brick to change the sign on this terminal $-x$ to x. At this point we have accounted everything in the definition of ξ_{14} and (3.11). The weighted sum over all such brick tabloids decorated in this way is equal to $\xi_{14}(h_n)$.

Scan the bricks from left to right looking for a $-x$ or two consecutive bricks which may be combined to preserve the ordering of the pictures. Just as in the proof of Theorem 3.10, break or combine the bricks accordingly, changing the power on x in the process. This involution is sign-reversing and weight-preserving. The fixed points cannot have any $-x$ labels and hence all bricks must contain one and only one picture. Furthermore, in a fixed point, the pictures when read from left to right must be written in increasing order. For example, in the case of taking $\mathcal{P}_n$ as the set of pictures of unlabeled trees, one possible fixed point is found below.



These fixed points correspond to objects in $\mathcal{U}_n$ with powers of x counting the number of pictures. Summing over all fixed points, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} t^n \sum_{s\in\mathcal{U}_n} x^{\mathrm{pic}(s)} \\
&= \xi_{14}\left(\sum_{n=0}^{\infty} h_n t^n\right) \\
&= \left(\sum_{n=0}^{\infty} (-t)^n(-1)^n \sum_{i_1+2i_2+\cdots+ni_n=n} \binom{|\mathcal{P}_1|}{i_1}\cdots\binom{|\mathcal{P}_n|}{i_n}(-x)^{i_1+\cdots+i_n}\right)^{-1} \\
&= \left(\prod_{i=1}^{\infty}\sum_{j=0}^{\infty}\binom{|\mathcal{P}_j|}{j}(-x)^j(t^i)^j\right)^{-1}
\end{aligned}$$

which, by an application of the binomial theorem, is equal to the desired expression. $\square$

Let us give an example of how Theorem 3.11 may be used to find a well known generating function for the number of partitions of n refined by length. Let $\mathcal{P}_n$ be a set with one picture, the picture consisting of a horizontal strip of n cells. Elements in $\mathcal{U}_n$ are composite objects built from these horizontal strips where the total number of cells is n. These composite objects may be considered Ferrers' diagrams for partitions. The number of pictures used to build a partition is its length. Therefore,

$$\sum_{n=0}^{\infty} t^n \sum_{\lambda\vdash n} x^{\ell(\lambda)} = \prod_{i\geq 1}\frac{1}{(1-xt^i)}.$$

If one so desired, one may find a generating function refining partitions by the parts it contains to find a partition analogue of the cycle index polynomial.

The proof we have given for Theorem 3.11—and the proof of Theorem 3.10 for that matter—is not the most direct proof possible (although it is elementary and combinatorial). As is the case for most of the theorems in this thesis, we have proved this theorem in the manner above to further demonstrate the versatility of the one method we are developing.

Jacobi indicated a formal version of the exponential formula and special cases of the exponential formula were given for permutations by Touchard in 1939 and graphs by Riddell and Uhlenbeck in 1953 [**46, 62, 69**]. The full generality we give in Theorem 3.10 and Theorem 3.11 was first published in the early 1970's in papers by Bender and Goldman, Doubilet with Rota and Stanley, and Foata and Schützenberger [**10, 28, 39**]. Since that time there have been a number of extensions of the theory [**11, 8, 41, 44, 47**].

In this section we will use the ability to weight the last brick in a brick tabloid differently than the others to further refine composite objects built from pictures with n circles.

Given an element $s \in \mathcal{L}_n$ built with components found in $\mathcal{P}_1, \ldots, \mathcal{P}_n$, let $\mathrm{one}(s)$ be the size of the component in which the label 1 may be found. For example, if s is the set partition

$$\{9\}, \{4, 12\}, \{2, 5, 7, 10, 11\}, \{1, 3, 6, 8\}$$

of 12 (as defined in Section 3.4), then $\mathrm{one}(s) = 4$ because the number 1 appears within a set of size 4.

To refine Theorem 3.10 by the place where 1 appears, let $j \geq 1$ and define the function to weight the last brick ν_6 such that

$$\nu_6(n) = \begin{cases} 0 & \text{if } n \neq j \\ \dfrac{(-1)^j}{(j-1)!} \dfrac{(-x)|\mathcal{P}_j|}{\xi_{13}(e_j)} & \text{if } n = j. \end{cases} \tag{3.12}$$

where ξ_{13} is the homomorphism defined in Section 3.4.

THEOREM 3.12.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{s \in \mathcal{L}_n} x^{\mathrm{pic}(s)} z^{\mathrm{one}(s)} = \int x \mathbf{e}^{x\mathcal{P}(t)} \frac{d}{dt} \left(\mathcal{P}(zt)\right) \, dt$$

where $\mathcal{P}(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} |\mathcal{P}_n|$ is the generating function for the number of pictures with n circles.

PROOF. In the proof of Theorem 3.10, the factor of $(-1)^i/i!$ in the definition of ξ_{13} gave rise to the -1 weight and subset of the first n integers of size i associated to each brick. The rest of the factors in the definition of ξ_{13} filled the bricks with pictures sorted in a particular manner and a weight of $-x$ for each picture.

In this situation, we have by expanding p_{n,ν_6} in terms of the elementary basis,

$$(n-1)!\xi_{13}(p_{n,\nu_6}) = (n-1)! \sum_{\lambda \vdash n} \sum_{\substack{T \in B_{\lambda,(n)} \text{ has} \\ \text{bricks } b_1, \ldots, b_\ell}} (-1)^{n-\ell} \xi_{13}(e_{b_1}) \cdots \xi_{13}(e_{b_\ell}) \nu_6(b_\ell). \tag{3.13}$$

First of all, by the definition of ν_6, we only need to consider brick tabloids where the final brick has length j. Just as in the proof of Theorem 3.10, we have a factor of $(-1)^i/i!$ for each brick of size i—except for the final brick for which we have a factor of $(-1)^i/(i-1)!$. This changed factor for the last brick comes from the definition of ν_6 which may be used to cancel the factor $\xi_{13}(e_{b_\ell})$ in (3.13) and replace it with $\frac{(-1)^j}{(j-1)!}(-x)|\mathcal{P}_j|$. Therefore, implicit in (3.13), there is a binomial coefficient of the form $\binom{n-1}{b_1, \ldots, b_\ell - 1, j-1}$ for each brick tabloid T. With this binomial coefficient, select a subset of the first n integers of size i for each brick of size i while forcing the integer 1 to be in the subset of size j assigned to the final brick in a brick tabloid (which also must be of size j).

The powers of -1 in (3.13) are the same as found in the proof of Theorem 3.10. Furthermore, except for the final brick, pictures may be inserted in the brick tabloid just as in the proof of Theorem 3.10. The last brick, as dictated by ν_6, should contain only one picture from $\mathcal{P}_j$ along with one x.

In conclusion, the application of ξ_{13} on $(n-1)!p_{n,\nu_6}$ produces brick tabloids filled with pictures and powers of x and $-x$ just like those found in the proof of theorem with the following exceptions: the last brick must be of size j and contain only one picture and the last brick must contain the circle labeled 1.

The same involution as in Theorem 3.10 follows through in this case. Notice that since the 1 label must appear in the last brick and that brick contains only one picture, we would never want to combine the last two bricks in any object we have built. Fixed points under this involution correspond to elements in $\mathcal{L}_n$ with the 1 appearing in a picture of size j. The x weights on a fixed point correspond to the number of pictures used to build the object. Therefore, we have

$$\begin{aligned}
\frac{d}{dt}\left(\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{s\in\mathcal{L}_n} x^{\mathrm{pic}(s)}z^{\mathrm{one}(s)}\right) &= \sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}\sum_{s\in\mathcal{L}_n} x^{\mathrm{pic}(s)}z^{\mathrm{one}(s)}\\
&= \sum_{j=1}^{\infty}\frac{z^j}{t}\xi_{13}\left(\sum_{n=1}^{\infty}p_{n,\nu_6}t^n\right)\\
&= \sum_{j=1}^{\infty} z^j\frac{\sum_{n=1}^{\infty}(-1)^{n-1}\nu_6(n)\xi_{13}(e_n)t^{n-1}}{\sum_{n=0}^{\infty}(-t)^n\xi_{13}(e_n)}.
\end{aligned}$$

The denominator of the above expression was shown in Theorem 3.10 to equal $\mathbf{e}^{-x\mathcal{P}(t)}$. This, along with the definition of ν_6, shows that the above string of inequalities is equal to

$$\mathbf{e}^{x\mathcal{P}(t)}\sum_{j=1}^{\infty}z^j x|\mathcal{P}_j|\frac{t^{j-1}}{(j-1)!} = x\mathbf{e}^{x\mathcal{P}(t)}\frac{d}{dt}\left(\sum_{j=1}^{\infty}\frac{(zt)^j}{j!}|\mathcal{P}_j|\right) = x\mathbf{e}^{x\mathcal{P}(t)}\frac{d}{dt}\left(\mathcal{P}(zt)\right).$$

This string of inequalities began by taking the derivative operator on the desired generating function; applying the integral operator $\int\cdot\,dt$ to the extremities of this string of inequalities proves the theorem. $\square$

We note that by our definition of the operator $\int\cdot\,dt$ on the ring of formal power series, the coefficient of t^0 in the integral of any generating function is equal to 0. Also we note that in taking $z=1$, the integral in Theorem 3.12 simplifies to $\mathbf{e}^{x\mathcal{P}(t)}$, thereby revealing the special case of Theorem 3.10.

Every single result that one can find using the exponential formula may be refined by keeping track of where the 1 appears (more generally, we may keep track of any one distinguished integer in a composite object). So, for example,

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\substack{s\text{ is a set}\\ \text{partition with }k\text{ parts}}} x^k z^{\mathrm{one}(s)} = \int xz\mathbf{e}^{x(\mathbf{e}^t-1)+zt}\,dt$$

refines the Stirling numbers in (3.8). An extra indeterminate z may be added to the cycle index polynomial to remember the length of the cycle containing 1 to find

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in S_n} z^{\mathrm{one}(\sigma)}q_1^{\mathrm{cyc}_1(\sigma)}q_2^{\mathrm{cyc}_1(\sigma)}\cdots = \int z\left(q_1+q_2(zt)+\cdots\right)\mathbf{e}^{q_1t+q_2t^2/2+\cdots}\,dt.$$

This equation may be specialized in many different ways to restrict the cycles which may appear in a permutation as indicated in Section 3.4. One such specialization may be found by taking $q_1 = q_2 = \cdots = x$ to find

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{cyc}(\sigma)} z^{\mathrm{one}(\sigma)} = \int \frac{xz}{(1-t)^x(1-zt)}\, dt. \tag{3.14}$$

A generating function refining graphs by counting the number of nodes in the connected component containing 1 can be found as well, building from (3.10).

In the proof of Theorem 3.10 in Section 3.4, we sorted pictures within each brick in increasing order according to the smallest labels. Tracing back the proof of Theorem 3.10, this particular way of sorting pictures within bricks was arbitrary. Any linear order would prove the theorem. We chose this particular ordering of pictures so that the last brick in a fixed point would contain the circle labeled with 1. That way, in this section, using a modification of weight on the last brick in a brick tabloid, we placed the picture with j circles containing a circle labeled with 1 in the last brick to give Theorem 3.12. Different linear orders of pictures combined with the capacity to change the weight on a last brick can refine composite objects by their largest and smallest components as found in Theorem 3.13 below.

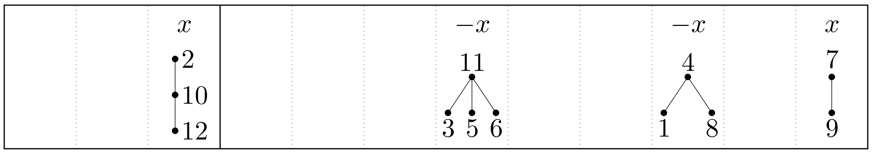
THEOREM 3.13.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{s \in \mathcal{L}_n} x^{\mathrm{pic}(s)} z^{\max(s)} \\
&= x\mathbf{e}^{x\mathcal{P}(t)} \left(\mathcal{P}(tz) + \sum_{j=1}^{\infty} z^j |\mathcal{P}_j| \right. \\
&\qquad \left. \times \sum_{n=j+1}^{\infty} \frac{t^n}{n!} \sum_{m=1}^{n-j} \sum_{\substack{i_1,\ldots,i_m \geq j \\ i_1+\cdots+i_m=n-j}} (-x)^m \binom{n}{i_1,\ldots,i_m,j} \frac{|\mathcal{P}_{i_1}|\cdots|\mathcal{P}_{i_m}|}{(m+1)!} \right)
\end{aligned}$$

where $\max(s)$ *is the size of picture with the maximum number of circles used in* s.

PROOF. The ideas in this proof are similar to those in the proof of Theorem 3.12 in the sense that the last brick in a brick tabloid will be modified to arrive at fixed points corresponding to composite objects s with $\max(s) = j$. To see what that particular last brick weight must be, let us first consider a reordering of pictures within bricks.

By the proof of Theorem 3.10, the application of ξ_{13} to $n!h_n$ gives bricks filled with pictures. Consider instead filling the bricks in a brick tabloid with pictures sorted in decreasing order first according to maximum size, then according to smallest labels. For example, when taking pictures of trees, the brick tabloid on page 115 filled with pictures written with respect to this new order may be found below.



By using the involution which breaks a brick at $-x$ or combines bricks when this new linear order on pictures is preserved, fixed points remain where the pictures when read left to right are in increasing order according to size. This means that if we wanted this last brick in a fixed point to contain a picture with j circles, we can force the last brick in a brick tabloid to have pictures with minimum size j. In other words, we may define a function to weight the last brick which maps n to 0 if $n < j$,

$$\frac{1}{\xi_{13}(e_j)}\frac{(-1)^j}{j!}(-x)|\mathcal{P}_j|$$

if $n = j$, and

$$\frac{1}{\xi_{13}(e_n)}\frac{(-1)^n}{n!}(-x)|\mathcal{P}_j|\sum_{m=1}^{n-j}\sum_{\substack{i_1,\dots,i_m\geq j\\ i_1+\dots+i_m=n-j}}(-x)^m\binom{n}{i_1,\dots,i_m,j}\frac{|\mathcal{P}_{i_1}|\cdots|\mathcal{P}_{i_m}|}{(m+1)!}$$

if $n > j$. This function will do a few things to the last brick. First, it will force the length of the last brick to be at least j cells; otherwise, the weight on the last brick is 0 and does not need to be considered.

If the last brick happens to have length j, then the normal labeling is canceled by the factor of the form $1/\xi_{13}(e_j)$. The term of the form $(-1)^n/(n!)$ will give a subset of the first n integers and a power of -1 to each brick as usual, and the $(-x)|P_j|$ term will place a picture with j circles in the last brick.

If the last brick has length greater than j, the normal labeling of the brick is canceled and replaced with a filling of pictures such that a picture with j circles must appear along with other pictures with minimum size j.

When the last brick is modified in this manner, fixed points corresponding to the usual involution are elements in $\mathcal{L}_n$ with maximum component equal to j. Therefore, by using the expansion of the new basis for Λ_n in terms of the elementary symmetric functions, the messy generating function in the statement of this theorem may be found. □

By sorting the pictures within a brick in increasing order and appropriately modifying the last brick in an analogous manner as in the above proof, a generating function refining the exponential formula according to the smallest picture can be found. Furthermore, modifying the last brick in a brick tabloid can refine the unlabeled version of the exponential formula in according to the largest and smallest pictures.

CHAPTER 4

Conclusions

We will end this monograph with a description of a couple of strategies on how to construct both unknown and already known generating functions using the ideas introduced in this thesis. This will serve as a synthesis of the techniques we have described.

There are advantages to finding a generating function from the point of view of brick tabloids and homomorphisms on the ring of symmetric functions as opposed to finding a generating function through recurrence equations or differential equations or other means. When a generating function is understood through brick tabloids, many times it may be generalized by many of the techniques laid out in the previous chapters of this thesis. Specifically, once a generating function is created in this way, there is potential that we may:

- specialize the result by forcing the powers of x and y on a brick tabloid to appear in specific patterns,
- extend the generating function for multiple objects or other permutation groups,
- add a number of q, p-analogues for inversions, coinversions, or major index type statistics, or
- use the last brick modification to further refine the generating function.

In building with brick tabloids, we gain an understanding of the object which may not be apparent through other means. Therefore, not only is it nice to find an unknown generating function through the idea of brick tabloids, but there is also a benefit from taking a known generating function and fitting it into our structure. This process is what we mean when we say "reverse engineering". As will appear evident, the biggest obstacle of either creating or reconstructing a generating function is to discover how bricks of length n should be weighted and filled.

We will explain the culmination of our techniques with a few examples. Let us start with an example where we are trying to discover a certain generating function. For $\sigma \in S_n$, let $\mathrm{drop}(\sigma)$ be the number of indices i between 1 and $n-1$ such that $\sigma_i = \sigma_{i+1} + 1$. Suppose we want to find a closed expression for

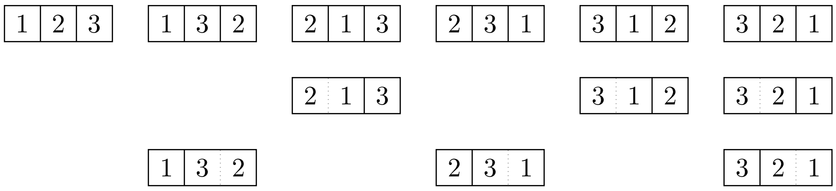
$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_n} x^{\mathrm{drop}(\sigma)} \tag{4.1}$$

Let us try to find out how to fill the bricks in a brick tabloid to end up with objects which register permutations along with powers of x corresponding to $\mathrm{drop}(\sigma)$. First, we will attempt to find a weighting function f which will tell us how to weight and label bricks. There are at least two different ways to gather information about this weighting function f.

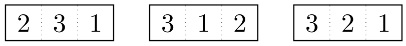
We may see how to fill a brick of length n with f inductively. That is, since our ultimate goal is to create a generating function for $\sum_{\sigma\in S_n} x^{\mathrm{drop}(\sigma)}$, we know that there should be one brick of length one filled with 1. This one brick of length one filled with 1 corresponds to the one element in S_1. Thus, a brick of length one should be weighted by the weighting function f with -1. That way, when combined with the one power of -1 assigned to each brick in the definition of $\mathcal{T}_f$, we have the desired one brick of length 1 with positive weight.

When we move on to bricks of length two, there are two different brick tabloids that may be formed using bricks of length one. These are the brick tabloids containing two bricks; one is filled with 1 and 2 while the other is filled with 2 and 1. Since both of these two brick tabloids have weight $+1$, this means that we would need to weight one brick of length 2 filled with 2 1 with $x-1$ to find $\sum_{\sigma\in S_2} x^{\mathrm{drop}(\sigma)}$. Thus, this means that $f(2)$ should equal $-(x-1)$.

Then, when we continue on to bricks of length 3, we already may form the bricks tabloids pictured below.



Each brick in the first row in the above figure is weighted with $+1$ while each of the bricks in the last two rows is weighted with $(x-1)$. This means that in order to find bricks we would need to weight the following bricks of length 3 in the figure below.



The above objects would need to have weights of $(1-x)$,$(1-x)$, and (x^2-2x+1) reading left to right. Therefore, accounting for the one power of -1 attached to each brick, it must be the case that $f(3) = -(x^2-4x+3)$.

Continuing on in this manner, one can find that the reciprocal to the desired generating function in (4.1) is equal to

$$1 - \frac{t^n}{1!} - \frac{(x-1)t^2}{2!} - \frac{(x^2-4x+3)t^3}{3!} - \frac{(x^3-11x^2+17x-7)t^4}{4!} - \cdots .$$

Now, if a pattern begins to emerge, then one can guess how to weight a brick of length n with the weighting function f and potentially prove the intended result. In this example, however, a pattern in the function f does not seem to be immediately clear.

There is another approach to finding this weighting function f in the example of trying to find a closed expression for (4.1). As indicated above, the value $f(n)$ will be $n!$ times the coefficient of t^n in the reciprocal of the generating function in (4.1). That is, we want to understand the coefficient of t^n in

$$\left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma\in S_n} x^{\mathrm{drop}(\sigma)}\right)^{-1} .$$

However, this entire monograph has been devoted to understanding the coefficient of t^n in reciprocals of this form. We may use what we have developed to find information about f by decorating brick tabloids with the weighting function $\sum_{\sigma\in S_n} x^{\mathrm{drop}(\sigma)}$. If this is done, we find the following bricks of length one, two, and three below.

-1
1

	-1
1	2

x	-1
2	1

		-1
1	2	3

	x	-1
1	3	2

x		-1
2	1	3

		-1
2	3	1

		-1
3	1	2

x	x	-1
3	2	1

The power of -1 above the last cell of each brick in this figure comes from our description of the set $\mathcal{T}_f$ found in Section 2.1. The powers of x are placed over cells which register a drop after them and these powers of x arise from using the weighting function $\sum_{\sigma\in S_n} x^{\mathrm{drop}(\sigma)}$.

These weighted bricks may be used to form brick tabloids. At this point, there is potential to find an involution to leave fixed points. Those fixed points will give the weighting function f. Then, the process may be reversed and the weighting function f may be used to decorate brick tabloids on which to perform involutions which leave fixed points corresponding to $\sum_{\sigma\in S_n} x^{\mathrm{drop}(\sigma)}$. Sometimes it will be the case that the actions to take in these involutions are obvious. In this example, however, it is not clear exactly what to do to find these involutions.

Even though the previous two attempts at finding a weighting function to provide a closed expression for (4.1) did not immediately pan out, all hope is not lost. There are still ways to find a generating function for (4.1). We have not yet used the idea of weighting the last brick in a brick tabloid differently than the others. This is what we will do in the proof of Theorem 4.1 below.

Theorem 4.1.

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma\in S_n} x^{\mathrm{drop}(\sigma)} = 1 + \int \frac{\mathbf{e}^{t(x-1)}}{(1-t)^2}\, dt.$$

Proof. First we will describe how to fill a brick of length k which does not appear last in a brick tabloid. Place the subset of $\{1,\ldots,n\}$ assigned to each brick in decreasing order. Then, choose to weight every cell in the brick with either 1 or $-x$. With the power of -1 given to each brick, reverse the power on the terminal 1 or $-x$.

The brick of length k at the end of the brick tabloid will be weighted in a different manner than the other bricks in a brick tabloid. If there are n total cells in the brick tabloid, add the integer $n+1$ to the subset of $\{1,\ldots,n\}$ assigned to the final brick. Permute these $k+1$ integers in any manner and place them into the cells of the final brick. Since there are only k cells in the final brick one of these integers may be placed to the right of the tabloid. Weight the last brick in a brick tabloid with all 1's—except in one special case. If the permutation in last brick happens to be the permutation which only decreases, it may be weighted with all 1's or we may place either a $-x$ or 1 above every integer except for $n+1$ and reverse the sign on the terminal $-x$ or 1. In a moment we will see why this special weighting is needed for this one particular filling.

A typical brick tabloid decorated in this manner may be found below.

1	$-x$	-1	$-x$	$-x$	x	1	x	x	1	1	1	1
11	10	2	8	5	1	4	3	6	7	13	9	12

We will apply two involutions to these objects. First, scan the brick tabloid from left to right looking for either

(1) a brick which does not end the brick tabloid and is of length greater than one, or
(2) a brick of length one followed by a brick which does not end the brick tabloid such that the two bricks straddle a descent in the integer labeling.

If when scanning, situation 1 on the above list appears first, break the first cell off of the brick and reverse the sign on the weight in the new brick of length one. If the second situation on the above list appears first, combine the two bricks into one and reverse the sign on the weight in the first cell of the brick. It is not difficult to see that this process is a sign-reversing weight-preserving involution. Fixed points may look like the brick tabloid found below.

-1	x	-1	x	x	x	-1	x	x	1	1	1	1
1	2	3	4	5	6	8	10	11	7	13	9	12

An example of a fixed point where the integers in the last brick are written in decreasing order may be found below.

-1	x	-1	x	x	x	-1	x	x	1	$-x$	1	x
1	2	3	4	5	6	8	10	11	13	12	9	7

Fixed points must consist of a string of increasing bricks of length one with weight either x or -1 followed by the terminal brick in the brick tabloid (which we have not yet touched).

A second sign-reversing involution will leave only objects with positive weight. This involution will be broken into two cases for clarity even though our actions are essentially the same in each case. First, let us consider fixed points under the first involution which do not have the integers in the final brick written in decreasing order. The involution in this case is the following. In a brick tabloid with n total cells, find the cell with the integer n. If the integer n appears in the final brick immediately after the cell marked with $n+1$ and the weight on the cell containing n is 1, do the following three things:

(1) remove the cell containing n from the last brick,
(2) place this brick of length one in with the sequence of increasing bricks preceding the final brick in the brick tabloid, and
(3) reverse the sign on the weight.

If the integer n appears in a brick of length one in the sequence of increasing bricks preceding the final brick and the weight on this brick is -1, then do the following:

(1) remove the brick of length one from the sequence of increasing bricks,
(2) add a cell after the brick of length $n+1$ and in it place the integer n, and
(3) reverse the sign on the weight.

If it so happens that the cell containing n is in the final brick but not in the cell immediately following the cell containing $n+1$ or if the cell containing n is in the sequences of increasing bricks with weight x, do nothing.

After considering the placement of the cell with n, move on to the cell containing $n-1$. Preform the same actions described above except this time examining the relative placement of the cells filled with $n-1$ and n. Continue on in this manner until, finally, the cells filled with 1 and 2 are examined. For example, the image of the second brick tabloid displayed in this proof may be found below.

-1	x	-1	x	x	x	x	x	1	1	1	1	1
1	2	3	4	5	6	10	11	7	13	9	8	12

In the second case of this involution we must consider fixed points under the first involution where the integers in the final brick are written in decreasing order. In this case, the involution is basically the same as the involution we have just described with one difference. In this involution, insert bricks whether or not they are weighted with a power of x or not. For example, the image of the fixed point given in the third figure in this proof may be found below.

-1	x	-1	x	x	x	-1	x	x	x	1	1	x
1	2	3	4	5	6	8	10	11	12	13	9	7

This insertion process of placing one cell into the final brick immediately following the next largest integer is a weight-preserving sign-reversing involution.

Fixed points under both involutions must consist of a sequence of bricks of length one followed by a final brick. The bricks of length one must be written in increasing order and weighted with powers of x. The final brick must contain a permutation $\sigma = \sigma_1 \cdots \sigma_k$ with the property that σ_i is never equal to $\sigma_{i+1}+1$. By inserting the integers in the sequence of bricks of length one, these fixed points may be seen to equal $\sum_{\sigma \in S_{n+1}} x^{\mathrm{drop}(\sigma)}$.

We weighted brick of length n which did not appear as the last brick in a brick tabloid with a power of $(1-x)^n$ to account for our choice of weight of 1 or $-x$ in each cell. As for the final brick in a brick tabloid, there are $(n+1)!-1$ ways to permute the $n+1$ integers in the final brick as to not place them in decreasing order. If we happen to write the integers in the last brick in decreasing order then we have $1-(1-x)^n$ choices for the weighting described above. Due to our discussion concerning weighting the last brick in a brick tabloid differently than the rest of the bricks, this means that

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in S_{n+1}} x^{\mathrm{drop}(\sigma)} = \frac{\sum_{n=1}^{\infty} \frac{t^n}{n!}\left((n+1)! - (1-x)^n\right)}{\sum_{n=0}^{\infty} \frac{t^n}{n!}(1-x)^n} = \frac{(1-t)^{-2} - \mathbf{e}^{t(1-x)}}{\mathbf{e}^{t(1-x)}}.$$

Adding 1, integrating to match the coefficient of t^n with the sum over the symmetric group, and simplifying, we arrive at the statement of the theorem. □

The point of including this proof in this final chapter is to demonstrate three different techniques to creating a generating function for a given statistic over the symmetric group. First, we tried to build up the bricks we needed inductively. Second, we tried filling brick tabloids using the weighting function $\sum_{\sigma \in S_n} x^{\mathrm{drop}(\sigma)}$ and then attempted to find an involution which left fixed points corresponding to the weighting function we needed. In both of these situations, patterns did not seem apparent.

Therefore we showed in the proof of Theorem 4.1 how to use the last brick weight in a brick tabloid to find the desired generating function. Since the involution in the proof of Theorem 4.1 rearranges the integer in the permutation, p, q-analogues are not immediate. Further, since we weighted the last brick differently than the others, we are not able to use the techniques of changing the last brick in a brick tabloid to extend the result. We note that if we were able to find the weighting function f in this case, then there is a much greater chance of using this machinery we have built up to generalize the generating function.

If we know what the generating function for a given sequence actually is, then the process of reverse engineering becomes a little easier. We say that a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is 3 2 1-avoiding provided there are not indices $i < j < k$ such that $\sigma_i > \sigma_j > \sigma_k$. It is known that the number of 3 2 1-avoiding permutations of n is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. Therefore, we can find the weighting function f in this case. It is given by

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{1}{n+1}\binom{2n}{n}\right)^{-1}\\ &\qquad = 1 - \frac{t^1}{1!} - \frac{-t^3}{3!} - \frac{-2t^4}{4!} - \frac{2t^5}{5!} - \frac{28t^6}{6!} - \frac{65t^7}{7!} - \frac{-338t^8}{8!} - \frac{-3262t^9}{9!} - \cdots .\end{aligned}$$

This means that the total weights of bricks of length one should be 1, the total weights of bricks of length two should be 0, the total weights of bricks of length three should be -1, the total weights of bricks of length four should be -2, and so on.

This does not mean that there should be two bricks of length 4 with weight -1, however. If we build up the bricks we will need inductively as we did to begin this chapter in the example of the statistic counting the number of times a permutation σ has $\sigma_i = \sigma_{i+1} + 1$, then it may be seen that there should be 4 bricks of length four: three bricks should be filled with 3 2 4 1, 4 1 3 2, and 4 2 3 1 and weighted with -1 while one brick should be filled with 4 3 2 1 and weighted with $+1$ to give total weight to all the bricks of length four equal to -2 as indicated in the above series.

In this example, even though we know the generating function and therefore its reciprocal, it is not known how to prove that it counts the desired objects using brick tabloids. If this can be done there is a good chance that 3 2 1-avoiding permutations will be better understood. For now, however, this remains an open problem.

It is important to note that absolutely any generating function has the potential to be found in the methods we describe. After all, the methods in this monograph are concerned with the coefficient of t^n in the reciprocal of the desired generating function and every generating function has a reciprocal. From the examples we just outlined, filling and weighting bricks is a nontrivial activity, but once this is done, generating functions may be built brick by brick.

APPENDIX A

Permutation statistics

This appendix contains a listing of the permutation statistics used in this document for quick reference. In the descriptions of the below statistics, let $\sigma = \sigma_1 \cdots \sigma_n$ be an element of the symmetric group S_n.

Statistic	*Description*	*Page*
blk	number of one or more consecutive letters in a word	104
cdes	number of i such that $\sigma_i > \sigma_{i+1}$ and wreath product group elements paired with σ_i and σ_{i+1} are in the same conjugacy class	97
coinv	number of pairs $i < j$ such that $\sigma_i < \sigma_j$	36
coinv_B	number of pairs $i < j$ such that $\sigma_i <_\Theta \sigma_j$ (for $\sigma \in B_n$)	99
comaj	sum of i where $\sigma_i < \sigma_{i+1}$	39
comaj_B	co-major index for B_n, sum of i where $\sigma_i <_\Theta \sigma_{i+1}$	99
comdes	number of i such that $\sigma_i^j > \sigma_{i+1}^j$ for all j (for multiples of permutations)	35
cris	number of i such that $\sigma_i < \sigma_{i+1}$ or group elements paired with σ_i and σ_{i+1} are not in the same conjugacy class	98
cyc	number of cycles in a permutation	117
cyc_i	number of cycles of size i in a permutation	116
des	number of i such that $\sigma_i > \sigma_{i+1}$	5
des_B	number of i such that $\sigma_{i+1} <_\Theta \sigma_i$ (for $\sigma \in B_n$)	99
drop	the number of indices i between 1 and $n-1$ such that $\sigma_i = \sigma_{i+1} + 1$	125
exc	number of i such that $i > \sigma_i$	5
fd	length of final decreasing sequence	45
fin	length of final increasing sequence	47
fxd	number of i such that $\sigma_i = i$	27
inv	number of pairs $i < j$ such that $\sigma_i > \sigma_j$	5
inv_B	number of pairs $i < j$ such that $\sigma_j <_\Theta \sigma_i$ (for $\sigma \in B_n$)	99
jdes	number of i such that $\sigma_i > \cdots > \sigma_{i+j}$	63
jris	number of i such that $\sigma_i > \cdots > \sigma_{i+j}$ is false	63
maj	sum of i where $\sigma_i > \sigma_{i+1}$	5
maj_B	sum of i where $\sigma_{i+1} <_\Theta \sigma_i$ (for $\sigma \in B_n$)	99
max	the picture with the maximum number of circles (in reference to the exponential formula)	122
neg	number of integers paired with -1 (for elements in B_n)	98
one	size of picture containing 1 (in reference to the exponential formula)	120

Statistic	*Description*	*Page*
oneris	number of i such that there exists a j such that $\sigma_i^j < \sigma_{i+1}^j$ (for multiples of permutations)	35
pic	number of pictures used to build one object (in reference to the exponential formula)	114
pos	number of integers paired with $+1$ (for $\sigma \in B_n$)	98
ris	number of i such that $\sigma_i < \sigma_{i+1}$	22
ris_B	number of i such that $\sigma_{i+1} <_\Theta \sigma_i$ is not true (for $\sigma \in B_n$)	99
rlcomaj	sum of $n - i$ where $\sigma_i < \sigma_{i+1}$	39
rlmaj	sum of $n - i$ where $\sigma_i > \sigma_{i+1}$	39
τ-nlap	the maximum number of nonoverlapping τ-matches	68
τ-mch	τ-matches in the permutation σ	67
val	number of i such that $\sigma_i > \sigma_{i+1}$ and $\sigma_{i+1} < \sigma_{i+2}$	66
wdes	number of i such that $\sigma_i \geq \sigma_{i+1}$	102

Bibliography

[1] Désiré André. Developpements de sec x et de tang x. *C. R. Acad. Sci. Paris*, 88:965–967, 1879.

[2] Désiré André. Mémoire sur les permutations alternées. *J. Math. pur. appl.*, 7:167–184, 1881.

[3] George E. Andrews. *The theory of partitions*. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 2.

[4] Desiree Beck. *Permutation enumeration of the symmetric and hyperoctahedral groups and the combinatorics of symmetric functions*. PhD thesis, University of California, San Diego, 1993.

[5] Desiree Beck. The combinatorics of symmetric functions and permutation enumeration of the hyperoctahedral group. *Discrete Math.*, 163(1-3):13–45, 1997.

[6] Desiree Beck and Jeffrey Remmel. Permutation enumeration of the symmetric group and the combinatorics of symmetric functions. *J. Combin. Theory Ser. A*, 72(1):1–49, 1995.

[7] Desiree Beck, Jeffrey Remmel, and Tamsen Whitehead. The combinatorics of transition matrices between the bases of the symmetric functions and the B_n analogues. In *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993)*, volume 153, pages 3–27, 1996.

[8] Janet S. Beissinger. *Factorization and enumeration of labeled combinatorial objects*. PhD thesis, University of Pennsylvania, 1981.

[9] Edward Bender and Donald Knuth. Enumeration of plane partitions. *J. Combinatorial Theory Ser. A*, 13:40–54, 1972.

[10] Edward A. Bender and Jay R. Goldman. Enumerative uses of generating functions. *Indiana Univ. Math. J.*, 20:753–765, 1970/1971.

[11] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota.

[12] Miklós Bóna. *Combinatorics of permutations*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2004. With a foreword by Richard Stanley.

[13] Francesco Brenti. Unimodal polynomials arising from symmetric functions. *Proc. Amer. Math. Soc.*, 108(4):1133–1141, 1990.

[14] Francesco Brenti. Permutation enumeration symmetric functions, and unimodality. *Pacific J. Math.*, 157(1):1–28, 1993.

[15] Alexander Burstein. *Enumeration of words with forbidden patterns*. PhD thesis, University of Pennsylvania, 1998.

[16] Alexander Burstein and Toufik Mansour. Words restricted by patterns with at most 2 distinct letters. *Electron. J. Combin.*, 9(2):Research paper 3, 14 pp. (electronic), 2002/03. Permutation patterns (Otago, 2003).

[17] Alexander Burstein and Toufik Mansour. Counting occurrences of some subword patterns. *Discrete Math. Theor. Comput. Sci.*, 6(1):1–11 (electronic), 2003.

[18] Alexander Burstein and Toufik Mansour. Words restricted by 3-letter generalized multipermutation patterns. *Ann. Comb.*, 7(1):1–14, 2003.

[19] Leonard Carlitz. Sequences and inversions. *Duke Math. J.*, 37:193–198, 1970.

[20] Leonard Carlitz. Enumeration of up-down permutations by number of rises. *Pacific J. Math.*, 45:49–58, 1973.

[21] Leonard Carlitz. Permutations and sequences. *Advances in Math.*, 14:92–120, 1974.

[22] Leonard Carlitz. Permutations, sequences and special functions. *SIAM Rev.*, 17:298–322, 1975.

[23] Leonard Carlitz and Richard Scoville. Enumeration of permutations by rises, falls, rising maxima and falling maxima. *Acta Math. Acad. Sci. Hungar.*, 25:269–277, 1974.

[24] Leonard Carlitz and Richard Scoville. Generalized Eulerian numbers: combinatorial applications. *J. Reine Angew. Math.*, 265:110–137, 1974.

[25] Leonard Carlitz and Richard Scoville. Enumeration of up-down permutations by upper records. *Monatsh. Math.*, 79:3–12, 1975.

[26] Leonard Carlitz, Richard Scoville, and Theresa Vaughan. Enumeration of pairs of permutations. *Discrete Math.*, 14(3):215–239, 1976.

[27] Chak-On Chow. On the Eulerian polynomials of type D. *European J. Combin.*, 24(4):391–408, 2003.

[28] Peter Doubilet, Gian-Carlo Rota, and Richard Stanley. On the foundations of combinatorial theory. VI. The idea of generating function. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pages 267–318, Berkeley, Calif., 1972. Univ. California Press.

[29] Ömer Eğecioğlu and Jeffrey Remmel. Brick tabloids and the connection matrices between bases of symmetric functions. *Discrete Appl. Math.*, 34(1-3):107–120, 1991. Combinatorics and theoretical computer science (Washington, DC, 1989).

[30] Sergi Elizalde and Marc Noy. Consecutive patterns in permutations. *Adv. in Appl. Math.*, 30(1-2):110–125, 2003. Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001).

[31] Roger Entringer. A combinatorial interpretation of the Euler and Bernoulli numbers. *Nieuw Arch. Wisk. (3)*, 14:241–246, 1966.

[32] Roger Entringer. Enumeration of permutations of $(1, \cdots, n)$ by number of maxima. *Duke Math. J.*, 36:575–579, 1969.

[33] Leonhard Euler. *Foundations of differential calculus*. Springer-Verlag, New York, 2000. Translated from the Latin by John D. Blanton.

[34] Jean-Marc Fédou and Don Rawlings. More statistics on permutation pairs. *Electron. J. Combin.*, 1:Research Paper 11, approx. 17 pp. (electronic), 1994.

[35] Jean-Marc Fédou and Don Rawlings. Statistics on pairs of permutations. *Discrete Math.*, 143(1-3):31–45, 1995.

[36] Dominique Foata. On the Netto inversion number of a sequence. *Proc. Amer. Math. Soc.*, 19:236–240, 1968.

[37] Dominique Foata and Guo-Niu Han. Calcul basique des permutations signées. I. Longueur et nombre d'inversions. *Adv. in Appl. Math.*, 18(4):489–509, 1997.

[38] Dominique Foata and Guo-Niu Han. Calcul basique des permutations signées. II. Analogues finis des fonctions de Bessel. *Electron. J. Combin.*, 4(2):Research Paper 9, approx. 25 pp. (electronic), 1997. The Wilf Festschrift (Philadelphia, PA, 1996).

[39] Dominique Foata and Marcel-Paul Schützenberger. *Théorie géométrique des polynômes eulériens*. Lecture Notes in Mathematics, Vol. 138. Springer-Verlag, Berlin, 1970.

[40] Adriano Garsia and Ira Gessel. Permutation statistics and partitions. *Adv. in Math.*, 31(3):288–305, 1979.

[41] Adriano Garsia and Saj-Nicole Joni. Composition sequences. *Comm. Algebra*, 8(13):1195–1266, 1980.

[42] Ira Gessel. *Generating functions and enumeration of sequences*. PhD thesis, Massachusetts Institute of Technology, 1977.

[43] Ian Goulden and David Jackson. *Combinatorial enumeration*. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1983. With a foreword by Gian-Carlo Rota, Wiley-Interscience Series in Discrete Mathematics.

[44] Michael Henle. Dissection of generating functions. *Studies in Appl. Math.*, 51:397–410, 1972.

[45] David Jackson and Romas Aleliunas. Decomposition based generating functions for sequences. *Canad. J. Math.*, 29(5):971–1009, 1977.

[46] Carl Gustav Jacobi. Zur combinatorischen Analysis. *J. Reine Ang. Math.*, 22:372–374, 1841.

[47] André Joyal. Une théorie combinatoire des séries formelles. *Adv. in Math.*, 42(1):1–82, 1981.

[48] Sergey Kitaev. Partially ordered generalized patterns. to appear in Discrete Math.

[49] Sergey Kitaev. *Generalized patterns in words and permutations*. PhD thesis, Chalmers University of Technology and Göteborg University, 2003.

[50] Sergey Kitaev and Toufik Mansour. Partially ordered generalized patterns and k-ary words. *Ann. Comb.*, 7(2):191–200, 2003.

[51] Thomas Langley. *The plethysm of two Schur functions at hook, near-hook, and two row shapes*. PhD thesis, University of California, San Diego, 2001.

[52] Thomas Langley. Alternate transition matrices for Brenti's q-symmetric functions and a class of (q,t)-symmetric functions on the hyperoctahedral group. In *Proceedings of Formal power series and algebraic combinatorics (FPSAC)*, Melbourne, Australia, 2002.

[53] Thomas Langley and Jeffrey Remmel. Enumeration of m-tuples of permutations and a new class of power bases for the space of symmetric functions. submitted to Elsevier Science.

[54] Ian Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[55] Percy A. MacMahon. *Combinatory analysis*. Two volumes (bound as one). Chelsea Publishing Co., New York, 1960.

[56] Anthony Mendes, Jeffrey Remmel, and Jennifer Wagner. A λ-ring Frobenius characteristic for $G \wr S_n$. submitted to the Electronic Journal of Combinatorics.

[57] Eugen Netto. *Lehrbuch der Combinatorik*. Chelsea Publishing Company, New York, 1958.

[58] Arun Ram, Jeffrey B. Remmel, and Tamsen Whitehead. Combinatorics of the q-basis of symmetric functions. *J. Combin. Theory Ser. A*, 76(2):231–271, 1996.

[59] Victor Reiner. Signed permutation statistics. *European J. Combin.*, 14(6):553–567, 1993.

[60] Victor Reiner. Signed permutation statistics and cycle type. *European J. Combin.*, 14(6):569–579, 1993.

[61] Victor Reiner. Upper binomial posets and signed permutation statistics. *European J. Combin.*, 14(6):581–588, 1993.

[62] Robert Riddell and George Uhlenbeck. On the theory of the virial development of the equation of state of monoatomic gases. *J. Chem. Phys.*, 21:2056–2064, 1953.

[63] Theodore Rivlin. *Chebyshev polynomials*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1990. From approximation theory to algebra and number theory.

[64] Bruce Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[65] Richard P. Stanley. Binomial posets, Möbius inversion, and permutation enumeration. *J. Combinatorial Theory Ser. A*, 20(3):336–356, 1976.

[66] Richard P. Stanley. *Enumerative combinatorics. Vol. I*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986. With a foreword by Gian-Carlo Rota.

[67] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[68] James Joseph Sylvester. Note on the numbers of Bernoulli and Euler and a new theorem concerning prime numbers. In *The Collected Mathematical Papers of James Joseph Sylvester*, volume 2. Chelsea, New York, 1973.

[69] Jacques Touchard. Sur les cycles des substitutions. *Acat. Math.*, 70:243–297, 1939.

[70] Jennifer Wagner. *The combinatorics of the permutation enumeration of wreath products between cyclic and symmetric groups*. PhD thesis, University of California, San Diego, 2000.

[71] Jennifer Wagner. The permutation enumeration of wreath products $C_k \S S_n$ of cyclic and symmetric groups. *Adv. in Appl. Math.*, 30(1-2):343–368, 2003. Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001).