A Fundamental Region for Hecke's Modular Group

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Hecke proved analytically that when \( h > 2 \) or when \( h = 2 \cos(\pi/q), q \in \mathbb{Z}, q > 3, \) then \( B(h) = \{ \tau: \text{Im} \tau > 0, |\text{Re} \tau| < \lambda/2, |\tau| > 1 \} \) is a fundamental region for the group \( G(h) = \langle S_A, T \rangle, \) where \( S_A: \tau \mapsto \tau + \lambda \) and \( T: \tau \mapsto -1/\tau. \)

He also showed that \( B(\lambda) \) fails to be a fundamental region for all other \( \lambda > 0 \) by proving that \( G(\lambda) \) is not discontinuous. We give an elementary proof of these facts and prove a related result concerning the distribution of \( G(\lambda) \)-equivalent points.

For each \( \lambda > 0, \) let \( G(\lambda) \) be the group generated by the transformations \( S_A: \tau \mapsto \tau + \lambda \) and \( T: \tau \mapsto -1/\tau \) defined on \( H = \{ \tau: \text{Im} \tau > 0 \}. \) Let \( B(\lambda) = \{ \tau \in H: |\text{Re} \tau| < \lambda/2, |\tau| > 1 \}. \) Let \( \mathbb{Z} \) denote the integers.

Hecke \([1, \text{pp. 11}-20; 2, \text{pp. 599}-616]\) proved analytically that \( B(\lambda) \) is a fundamental region (as defined in \([3, \text{p. 22}]\)) for \( G(\lambda) \) when \( h \geq 2 \) or when \( \lambda = 2 \cos(\pi/q) \) for some \( q \in \mathbb{Z}, q > 3 \) (in the latter case we write \( \lambda \in \mathbb{C} \)). We give an elementary proof of this fact. When \( 0 < \lambda < 2, \lambda \notin \mathbb{C}, \) Hecke \([2, \text{pp. 609, 613}-614]\) proved that \( G(\lambda) \) is not discontinuous (so that there can be no fundamental region for \( G(\lambda) \)). We present here a slightly simplified version of his proof and show, moreover, that for any \( \tau \in H, \) the set of all points \( G(\lambda) \)-equivalent to \( \tau \) is dense in \( H. \)

**Theorem 1.** Each \( \gamma \in H \) is \( G(\lambda) \)-equivalent to a point in \( \overline{B(\lambda)} \), (the closure of \( B(\lambda) \)).

**Proof.** Define the following transformations on \( H: \)

\[ T_1: \tau \mapsto \tau/|\tau|^2 \text{ (reflection in the unit circle)}, \]
\[ T_2: \tau \mapsto -\tau \text{ (reflection in the line Re } \tau = 0), \]
\[ T_3: \tau \mapsto -(\tau + \lambda) \text{ (reflection in the line Re } \tau = -\lambda/2). \]

Since \( S_A = T_2 T_3 \) and \( T = T_1 T_2, \) it is easily seen that \( G(\lambda) \) consists of the
words in $\langle T_1, T_2, T_3 \rangle$ of even length. Hence, it suffices to find $V \in \langle T_1, T_2, T_3 \rangle$ such that $V \gamma \in B(\lambda)$, if for $V \notin G(\lambda)$, then $T_2 V \in G(\lambda)$.

Define a sequence of points $\tau_n = x_n + iy_n$ inductively as follows: apply $T_2$ and $T_3$, if necessary, to move $\gamma$ horizontally to a point $\tau_1$ in the strip $E_\lambda = \{ \tau \in H: -\lambda/2 \leq \Re \tau \leq 0 \}$. Given $\tau_n (n \geq 1)$, apply $T_2$ and $T_3$ to move $T_1 \tau_n$ horizontally to a point $\tau_{n+1} \in E_\lambda$. We will assume that $| \tau_n | < 1$ for each $n$, otherwise the theorem is proved. Thus, $y_{n+1} = y_n / | \tau_n | ^2 > y_n$. Let $w$ be a cluster point of $\{ \tau_n \}$. Note $\Im w > 0$. If $| w | < 1$, then $\{ \tau_n \}$ has an infinite subsequence $\{ \tau_{n_k} \}$ such that $| \tau_{n_k} | \leq c < 1$, so that $y_{n_k} \geq y_n / c^{2(k-1)} \to \infty$ as $k \to \infty$, a contradiction. Hence, $| w | = 1$. When $\lambda < 2$, let $v$ denote the point of intersection between the unit circle and the line $\Re \tau = -\lambda/2$. We will assume that $\lambda < 2$ and that $w = v$ is the unique cluster point of $\{ \tau_n \}$, otherwise $T_1 \tau_n \in B(\lambda)$ for some large $n$. If $\arg \tau_n \leq \arg v$ for some $n$, then $\Im \tau_{n+1} > \Im v$, contradicting the fact that $y_{n+1} > y_n$. Let $n \geq N$. Note that $x_n < 0$, since $x_{n+1} = -\lambda - x_n / (x_n ^2 + y_n ^2)$. Let $n \geq N$. Letting $\pi \theta = \pi - \arg v$ (so that $\lambda = 2 \cos \pi \theta$), we have

$$x_{n+1} - x_n = \frac{1}{x_n} \left( -\lambda x_n - x_n ^2 + y_n ^2 - x_n ^2 \right) = -\frac{1}{x_n} \left( \lambda x_n + \cos ^2 (\arg \tau_n) + x_n ^2 \right) > -\frac{1}{x_n} \left( \lambda x_n + \cos ^2 (\arg v) + x_n ^2 \right) = -\frac{1}{x_n} \left( \lambda x_n + \cos \pi \theta ^2 \right) \geq 0.$$

Thus, $x_{n+1} > x_n$ for each $n \geq N$, which contradicts the fact that $x_n \to \Re v$.

Thus, $B(\lambda)$ is a fundamental region for $G(\lambda)$ if and only if no two distinct points of $B(\lambda)$ are $G(\lambda)$-equivalent. We now show this is the case when $\lambda \geq 2$ or $\lambda \in C$.

**Theorem 2.** When $\lambda \geq 2$, no two distinct points of $B(\lambda)$ are $G(\lambda)$-equivalent.

**Proof.** Choose $V \neq I$ (I is the identity) in $G(\lambda)$ and $\tau \in B(\lambda)$. We will show that $V \tau \notin B(\lambda)$. We can write $V$ in the form $V = S_k ^{r} S_{k-1} ^{r-1} \cdots S_0 ^{r} T S_k ^{r}$, where $r \geq 1$, each $k_i \in Z$, and $k_i \neq 0$ if $2 \leq i \leq r - 1$. Let $\tau_1 = \cdots$
\( TS_3^{k_1} TS_3^{k_2-1} \cdots TS_3^{k_r} \tau \). It is easily seen that \(| \tau_i | < 1 \) for \( 1 \leq i \leq r - 1 \). Thus, \( V_\tau = S_3^{k_r} \tau_1 \notin B(\lambda) \).

In order to handle the case \( \lambda \in C \), we shall need two lemmas. Whenever \( \lambda \in C \), we shall write \( \lambda = 2 \cos(\pi/q) \), where \( q \in \mathbb{Z}, q \geq 3 \).

**Lemma 1.** When \( \lambda \in C \), no two points of \( B(\lambda) \) are equivalent under a nonidentity transformation in \( \langle T_1, T_3 \rangle \).

**Proof.** If the lemma is false, then there exist points \( \tau, \tau' \in B(\lambda) \) with, say, \( \Im \tau' \geq \Im \tau \) and a word \( V \neq I \) in \( \langle T_1, T_3 \rangle \) such that \( V \tau = \tau' \). Note \( V \neq T_3 \), as \( T_3 \notin B(\lambda) \). Hence, as \( T_1 \) and \( T_3 \) have order 2, \( V \) can have either the form \( T_3^\alpha (T_1 T_3)^n \) or \( T_3^\alpha (T_3 T_1)^n \), where \( n \in \mathbb{Z}, n \neq 0 \), and \( \alpha = 0 \) or 1. If \( V \) has the latter form, then \( V = T_3^\alpha (T_1 T_3)^{-n} \) because \( T_3 T_1 = (T_1 T_3)^{-1} \). Thus, in any case \( V \) has the former form. Now for all \( n \in \mathbb{Z} \), \( (T_1 T_3)^n \) is the linear fractional transformation with matrix

\[
\begin{pmatrix}
\frac{a_n}{c_n} & \frac{b_n}{d_n}
\end{pmatrix} = \frac{1}{\sin \pi \theta} \begin{pmatrix}
\sin \pi \theta(1 - n) & -\sin \pi \theta n \\
\sin \pi \theta n & \sin \pi \theta(n + 1)
\end{pmatrix}
\]

Since \( (T_1 T_3)^n = I \), we may write \( V = T_3^\alpha (T_1 T_3)^n \), where \( \alpha = 0 \) or 1, \( n \in \mathbb{Z} \), \( 1 \leq n \leq q - 1 \). Write \( \tau = x + iy \). As \( c_n d_n \geq 0 \), we have

\[
|c_n \tau + d_n|^2 = c_n^2 |\tau|^2 + d_n^2 + 2c_n d_n x > c_n^2 + d_n^2 - \lambda c_n d_n = 1,
\]

so that

\[
\Im \tau' = \Im(T_1 T_3)^n \tau = \frac{y}{|c_n \tau + d_n|^2} < y = \Im \tau,
\]
a contradiction.

**Lemma 2.** Let \( \lambda \in C \), let \( x + iy = \tau \in H \), and let \( W \in \langle T_1, T_3 \rangle \), \( W \neq I \), \( W \neq T_1 \). If either

(i) \( \Re \tau > 0 \)

or

(ii) \( \tau \in B(\lambda) \),

then \( \Re W \tau < 0 \).

**Proof.** We can write \( W \) in the form \( W = T_3^\alpha (T_1 T_3)^n \), where \( \alpha = 0 \) or 1, \( n \in \mathbb{Z}, 1 \leq n \leq q - 1 \). To show that \( \Re W \tau < 0 \), it suffices to show that \( \Re(T_1 T_3)^n \tau < 0 \). We have (in the notation of the previous lemma)

\[
\Re(T_1 T_3)^n \tau = \frac{(a_n x + b_n)(c_n x + d_n) + a_n c_n y^2}{|c_n \tau + d_n|^2}.
\]
Note that \(a_n \leq 0\), \(b_n \leq 0\), \(c_n \geq 0\), and \(d_n \geq 0\). Hence, if (i) holds, \(a_n c_n y^2 \leq 0\) and \((a_n^2 + b_n)(c_n^2 + d_n) < 0\), so \(\text{Re}(T_1 T_3) \tau < 0\). If (ii) holds, then

\[
\text{Re}(T_1 T_3) \tau = \frac{b_n d_n + a_n c_n |\tau|^2 + (a_n d_n + b_n c_n) x}{c_n \tau + d_n |\tau|^2}
\]

\[
= \frac{b_n d_n + a_n c_n + (a_n d_n + b_n c_n)(-\lambda/2)}{c_n \tau + d_n |\tau|^2}
\]

\[
= \frac{-\cos(\pi/\lambda)}{c_n \tau + d_n |\tau|^2} < 0.
\]

**Theorem 3.** If \(\lambda \in \mathbb{C}\), no two distinct points of \(B(\lambda)\) are \(G(\lambda)\)-equivalent.

**Proof.** It suffices to show that no two points of \(B(\lambda)\) are equivalent under a transformation \(V \in \langle T_1, T_2, T_3 \rangle\), where \(V \neq T_1, V \neq T_2\). If the contrary is true, choose a word \(v\) for \(V\) in \(\langle T_1, T_2, T_3 \rangle\) of minimal length \(L\) for which \(V \neq T_2, V \neq T_1\), and there exists \(\tau \in B(\lambda)\) such that \(V \tau \in B(\lambda)\). By Lemma 1, such a word must contain \(T_2\). No word for \(V\) of length \(L\) can begin or end with \(T_2\). For if \(V = T_2 Y\), then \(Y \neq T_2, Y \neq T_1, Y \neq T_3\), and \(Y \tau \in B(\lambda)\), which contradicts the minimality of \(L\); similarly, if \(V = YT_2\), then \(Y \neq T_2, Y \neq T_1, Y \neq T_3\), and \(Y(T_2 \tau) \in B(\lambda)\), a contradiction. Thus, \(v = W_1 T_2 W_3 T_2 \cdots W_k T_2 W_{k+1} (k \geq 1)\), where \(I \neq W_i \in \langle T_1, T_3 \rangle\) for each \(i\). Moreover, for each \(i\), \(W_i \neq T_1\). For if \(W_i = T_1\) or \(W_{k+1} = T_1\), then \(T_1 T_2 = T_2 T_1\), \(V\) would equal a word of length \(L\) which begins or ends with \(T_2\); if \(W_i = T_1\) for some \(i\) such that \(2 \leq i \leq k\), then since \(T_2 T_1 = T_1, V\) would equal a word of length smaller than \(L\).

Let \(\tau_i = T_2 W_i T_2 W_{i+1} \cdots T_2 W_{k+1} \tau\). We will show by induction on \(i\) that \(\text{Re} \tau_i < 0\), \((2 \leq i \leq k + 1)\). Since \(V \tau \in B(\lambda)\), \(\text{Re} \tau_2 \neq \text{Re} W_{k+1}^{-1} \tau < 0\) by Lemma 2. Assume \(\text{Re} \tau_m < 0\) for an \(m\) such that \(2 \leq m \leq k\). Then \(\text{Re} T_2 \tau_m > 0\), so by Lemma 2, \(\text{Re} \tau_{m+1} \neq \text{Re} W_m^{-1} T_2 \tau_m < 0\), completing the induction. As \(\tau \in B(\lambda)\), \(\text{Re} W_{k+1} \tau < 0\) by Lemma 2. Hence, \(\text{Re} \tau_{k+1} = \text{Re} T_2 W_{k+1} \tau > 0\), a contradiction.

We now investigate the distribution of \(G(\lambda)\)-equivalent points in \(H\) when \(0 < \lambda < 2, \lambda \notin \mathbb{C}\).

**Lemma 3.** Let

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

be the matrix of the linear fractional transformation \(W \in G(\lambda)\). Then \(W\) has a fixed point in \(H\) if and only if \(|a + d| < 2\).
Proof. \( W\tau = \tau \) if and only if \( \tau = (a - d \pm \sqrt{(d + a)^2 - 4})/2c \).

**Lemma 4.** Suppose \( W \in G(\lambda) \) has infinite order and \( W \) has a fixed point \( \tau_1 \in H \). Let \( t(\tau) = (\tau - \tau_1)/(\tau - \bar{\tau}_1) \), where \( \bar{\tau}_1 \) is the complex conjugate of \( \tau_1 \). Then for each \( \tau \in H - \{\tau_1\} \), the set \( J_\tau = \{W^n\tau : n \in \mathbb{Z}\} \) is dense on the circle \( K_\tau = \{\sigma : |t(\sigma)| = |t(\tau)|\} \).

**Proof.** Let
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
be the matrix of \( W \). Note that \( \rho = c\tau_1 + d \) is the characteristic value of
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
corresponding to the characteristic vector \( \{\tau_1\} \).

Since \( \rho \) and \( \bar{\rho} \) are the roots of the characteristic equation
\[
x^2 - (a + d)x + 1 = 0,
\]
we have \( \rho \bar{\rho} = 1 \). Now for any \( \tau \),
\[
t(W\tau) = (W\tau - W\tau_1)/(W\tau - W\bar{\tau}_1)
\]
since \( \tau_1 \) and \( \bar{\tau}_1 \) are fixed by \( W \). Thus,
\[
t(W\tau) = \frac{\tau - \tau_1}{(c\tau + d)(c\tau_1 + d)} = \frac{\bar{\rho}}{\rho} t(\tau) = \rho^{-n} t(\tau).
\]
Thus, for all \( n \in \mathbb{Z} \), \( t(W^n\tau) = \rho^{-n} t(\tau) \). Since \( \tau_1 \) is nonreal and \( W \) has infinite order,
\[
\begin{pmatrix}
\tau_1 \\
1
\end{pmatrix} \neq \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^n \begin{pmatrix}
\tau_1 \\
1
\end{pmatrix} - \rho^n \begin{pmatrix}
\tau_1 \\
1
\end{pmatrix}, \text{ for each } n \geq 1.
\]
Otherwise, writing
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^n = \begin{pmatrix}
a^{(n)} & b^{(n)} \\
c^{(n)} & d^{(n)}
\end{pmatrix},
\]
we would have \( (a^{(n)} - 1)\tau_1 = -b^{(n)} \) and \( c^{(n)}\tau_1 = 1 - d^{(n)} \), so that \( a^{(n)} = d^{(n)} = 1 \) and \( b^{(n)} = c^{(n)} = 0 \), a contradiction.

Therefore, \( \rho \) is not a root of unity, and, consequently, \( \{t(W^n\tau) : n \in \mathbb{Z}\} \) is dense on the circle \( \{z : |z| = |t(\tau)|\} \). Thus, \( J_\tau \) is dense on \( K_\tau \).

**Lemma 5.** If \( 0 < \lambda < 2, \lambda \notin C \), then there exists a \( W \in G(\lambda) \) such that \( W \) has infinite order and \( W \) has a fixed point in \( H \).
Proof. Case 1. \( \theta \) is irrational. Choose \( W = TS_\lambda \) so that \( W \) has matrix

\[
\begin{pmatrix}
  a_1 & b_1 \\
  c_1 & d_1
\end{pmatrix}
\]

(in the notation of the proof of Lemma 1). By Lemma 3, \( W \) has a fixed point in \( H \). Since \( \theta \) is irrational, \( c_n \neq 0 \) for all \( n \geq 1 \). Thus \( W \) has infinite order.

Case 2. \( \theta = p/q \), \( (p, q) = 1 \), \( 2 \leq p < q/2 \). Choose \( W = T(TS_\lambda)^k \), where \( kp \equiv 1 \pmod{q} \). Note that \( W \) has matrix

\[
\begin{pmatrix}
  -c_k & -d_k \\
  a_k & b_k
\end{pmatrix} = \begin{pmatrix}
  -c_k & -d_k \\
  a_k & c_k
\end{pmatrix}.
\]

Since

\[
|c_k| = \left| \frac{\sin(\pi pk/q)}{\sin(\pi p/q)} \right| = \frac{\sin(\pi/q)}{\sin(\pi p/q)} < 1,
\]

\( W \) has a fixed point by Lemma 3.

To show that \( W \) has infinite order, we will show that

\[
\begin{pmatrix}
  -c_k & -d_k \\
  a_k & -c_k
\end{pmatrix}
\]

has a characteristic value \( \rho \) which is not a root of unity. Let \( c_k' \) be any algebraic conjugate of \( c_k \). Since \( \rho \) satisfies the characteristic equation \( x^2 + 2c_k'x + 1 = 0 \), a root \( \rho' \) of \( x^2 + 2c_k'x + 1 = 0 \) is a conjugate of \( \rho \).

When \( (j, 2q) = 1 \), \( (\sin(\pi pk/j))((\sin(\pi p/j)) \) is a conjugate of \( c_k \). If we let \( c_k' = (\sin(\pi pk/j)/\sin(\pi p/j)) \), where \( j \) is odd and \( jp \equiv 1 \pmod{q} \), then \( |c_k'| = |(\sin(\pi k/j))/\sin(\pi p/j)| \geq 1 \). Thus, \( \rho' \) is real. Now suppose \( \rho \) is a root of unity. Then so is \( \rho' \), so \( \rho' = \pm 1 \). Thus, \( \rho = \pm 1 \), which contradicts \( |c_k| < 1 \). Thus, \( \rho \) is not a root of unity.

It follows from Lemmas 4 and 5 that \( G(\lambda) \) is not discontinuous when \( 0 < \lambda < 2 \), \( \lambda \notin C \). We can prove a bit more.

**Theorem 4.** Let \( A(\tau) \) be the set of points which are \( G(\lambda) \)-equivalent to \( \tau \). If \( 0 < \lambda < 2 \), \( \lambda \notin C \), then for each \( \tau \in H \), \( A(\tau) \) is dense in \( H \).

**Proof.** By Lemma 5, we can find a \( W \in G(\lambda) \) such that \( W \) has infinite order and \( W \) has a fixed point \( \tau_1 \in H \). Define \( t(\tau) = (\tau - \tau_1)(\tau - \tau_2) \) as before. Assume there is a \( \tau \in H \) for which \( A(\tau) \) is not dense in \( H \). Then there is an open disk \( N \subset H - \{\tau_1\} \) such that \( N \cap A(\tau) = \emptyset \). If \( \sigma \in K_\alpha \cap A(\tau) \) for some \( \alpha \in N \), then \( N \) would contain a point in \( J_\sigma \) by Lemma 4, a contra-
diction. Thus, \( K_\alpha \cap A(\tau) = \emptyset \), for each \( \alpha \in \mathbb{N} \). We can, therefore, find \( e_1 \) and \( e_2 \) such that

\[
\{ \sigma \in A(\tau) : e_1 < |t(\sigma)| < e_2 \} = \emptyset.
\]

Let \( e_3 \) be the largest number for which \( \{ \sigma \in A(\tau) : e_1 < |t(\sigma)| < e_3 \} = \emptyset \). Note \( e_3 < 1 \), since \( |t(S_m \tau)| \to 1 \), as \( m \to \infty \). Define \( \beta \) to be the point with the largest real part satisfying \( |t(\beta)| = e_3 \). Note that \( \beta \) is the rightmost point on \( K_\beta \). The circles \( K_\beta \) and \( S_\lambda^{-1}K_{\beta+\lambda} \) intersect at \( \beta \) but they are not tangent because the center of \( S_\lambda^{-1}K_{\beta+\lambda} \) is higher than the center of \( K_\beta \). (The center of \( K_\beta \) is \( (x_1, y_1) = [(2/(1 - e_3^2)) - 1] \) and the center of \( K_{\beta+\lambda} \) is \( (x_1, y_1) = [(2/(1 - e_4^2)) - 1] \), where \( x_1 = x_1 + iy_1 \) and \( e_3 < e_4 = |t(\beta + \lambda)| < 1 \). By definition of \( e_3 \), there are points of \( A(\tau) \) arbitrarily close to \( K_\beta \). Hence, there are circles \( K_\lambda(\nu \in A(\tau)) \) in any small annulus containing \( K_\beta \). Lemma 4, thus, shows that \( \beta \) is a cluster point of \( A(\tau) \). Choose \( \mu \in A(\tau) \) so close to \( \beta \) that \( K_\beta \) and \( S_\lambda^{-1}K_{\mu+\lambda} \) intersect but are not tangent. Then there are points of \( S_\lambda^{-1}J_{\mu+\lambda} \) in \( \{ \sigma : e_1 < |t(\sigma)| < e_3 \} \), a contradiction. 

We conclude with some remarks concerning the distribution of \( G(\lambda) \)-fixed points in \( H \). A \( G(\lambda) \)-fixed point is a point in \( H \) fixed by some non-identity element of \( G(\lambda) \). When \( \lambda \geq 2 \) or \( \lambda \in C \), it is clear that \( B(\lambda) \) contains no \( G(\lambda) \)-fixed points. (For suppose \( V \tau = \tau \), where \( V \in G(\lambda) \), \( \tau \in B(\lambda) \). As \( V \) is continuous at \( \tau \), \( V \) maps a neighborhood \( N \) of \( \tau \) into \( B(\lambda) \). As no two distinct points of \( B(\lambda) \) are \( G(\lambda) \)-equivalent, \( V \) acts as the identity on \( N \). By the identity theorem, \( V = I \).)

The following corollary shows that the situation is quite different when \( 0 < \lambda < 2 \), \( \lambda \notin C \).

**Corollary.** If \( 0 < \lambda < 2 \), \( \lambda \notin C \), then the set \( F \) of \( G(\lambda) \)-fixed points is dense in \( H \).

**Proof.** Let \( \tau \in A(i) \), so that \( \tau = Vi \) for some \( V \in G(\lambda) \). Then \( VTV^{-1}\tau = \tau \), so \( \tau \in F \). Thus, \( A(i) \subset F \) and since \( A(i) \) is dense in \( H \) by Theorem 4, \( F \) is dense in \( H \).

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REFERENCES