MULTIDIMENSIONAL q-BETA INTEGRALS*

RONALD J. EVANS†

Abstract. A multidimensional extension of a q-beta integral of Andrews and Askey is evaluated. As an application, a short new proof of an important q-Selberg integral formula is given.

Key words. q-integral, Selberg integral, beta integrals

AMS(MOS) subject classification. 33A15

1. Introduction. This paper has been motivated by Anderson's wonderfully innovative proof [2] of Selberg's multidimensional beta integral formula [17]. In § 2 (see Theorem 1), we present a new n-dimensional q-beta integral formula which reduces to that of Andrews and Askey [4, eqn. (2.2)] when n = 1 and that of Anderson [2, "claim"] when q = 1. Our proof is self-contained and in particular makes no appeal to the results of the aforementioned papers. In § 3, we apply Theorem 1 to give a surprisingly short, self-contained proof of the q-Selberg integral formula (1.8). Finally, we indicate in § 4 the modifications that can be made in § 3 to give a short proof of Kadell's extension of the q-Selberg integral formula containing the extra parameter m of Aomoto [5]; see Theorem 2. It is hoped that this method will lead to a short proof of a q-extension of the Selberg-Jack integral formula [15].

For some of the many applications and extensions of Selberg's integral, see the papers of Askey [6]-[8] and Kadell [14]-[16]. For character sum analogues of Selberg's integral, see the papers of Anderson [1], Evans [10] and van Wamelen [18].

Let

\[ 0 < q < 1, \]

and define, for complex \( x, \alpha \),

\[ (\alpha)_x := \prod_{r=0}^{\infty} (1 - \alpha q^r), \quad (\alpha)_\infty := (\alpha)/(\alpha q^\infty). \]

Define the q-gamma function

\[ \Gamma_q(x) := (q)_{x-1}(1-q)^{1-x}, \quad x \in \mathbb{C}. \]

As \( q \to 1 \), \( \Gamma_q(x) \to \Gamma(x) \) [11, eqn. (1.10.3)]. For \( \alpha, \beta \in \mathbb{C} \) and a (say) continuous function \( f: \mathbb{C} \to \mathbb{C} \), define the q-integral

\[ \int_{\alpha}^{\beta} f(x) \, dq x := \int_{0}^{\beta} f(x) \, dq x - \int_{0}^{\alpha} f(x) \, dq x, \]

where

\[ \int_{0}^{\beta} f(x) \, dq x := (1-q) \sum_{m=0}^{\infty} f(\beta q^m) \beta q^m. \]

As \( q \to 1 \), \( \int_{\alpha}^{\beta} f(x) \, dq x \to \int_{\alpha}^{\beta} f(x) \, dx \) [11, p. 19]. For example, for \( m > 0 \),

\[ \int_{\alpha}^{\beta} x^{m-1} \, dq x = \frac{(\beta^m - \alpha^m)(1-q)}{(1-q^m)} \frac{\beta^m - \alpha^m}{m}. \]

* Received by the editors December 17, 1990; accepted for publication (in revised form) June 21, 1991.
† Department of Mathematics 0112, University of California, San Diego, La Jolla, California 92093-0112.
as \( q \to 1 \). The following \( q \)-integral extension of Euler's beta function integral is essentially a version of the \( q \)-binomial theorem [11, pp. 18–19]:

\[
(1.7) \quad \int_0^1 t^{a-1}(tq)_{b-1} \, dq = \Gamma_q(a)\Gamma_q(b)/\Gamma_q(a+b), \quad \text{Re} \,(a), \text{Re} \,(b) > 0.
\]

This is the case \( n = 1 \) of the following \( n \)-dimensional \( q \)-Selberg integral formula [13, eqn. (4.18)]:

\[
S_n(a, b, c) := \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{a-1}(tq)_{b-1} \prod_{i<j} (t_i - q^k t_j) \, dq_1 \cdots dq_n
\]

\[
(1.8) \quad = q^{ac(a) + bc(b) + c(c)} \prod_{j=0}^{n-1} \Gamma_q(a+jc)\Gamma_q(b+jc)\Gamma_q(c+jc)
\]

\[
= q^{ac(a) + bc(b) + c(c)} \prod_{j=0}^{n-1} \Gamma_q(a+b+(n-1+j)c)\Gamma_q(c),
\]

where \( n, c \) are positive integers and \( \text{Re} \,(a), \text{Re} \,(b) > 0 \). This reduces to Selberg’s integral formula [17] when \( q \to 1 \). Note that the integrand in (1.8) is symmetric in the variables \( t_i \). It is not difficult to show that the nonsymmetric version of (1.8) originally conjectured by Askey [6, Conj. 1] is equivalent to (1.8); see Kadell [13, p. 953]. Proofs of (1.8) have been given independently by Habsieger [12] and Kadell [14].

We observe here for later use that the value of the integral in (1.8) is unchanged if the upper limits of integration are replaced by \( q^{-u} \), when \( u \) and \( b \) are integers such that \( 0 \leq u \leq b-1 \). This is because \( (tq)_{b-1} \) vanishes for \( t = q^{-1}, q^{-2}, \ldots, q^{-u} \). It follows that the integral in (1.8) changes only by a factor of a power of \( q \) when the variables \( t_i \) are replaced by \( t_i q^{-u} \).

2. Extension of the Andrews–Askey \( q \)-integral.

**Theorem 1.** Let \( u_i, s_i \) be integers such that

\[
(2.1) \quad 0 \leq u_i \leq s_i - 1, \quad i = 0, 1, \ldots, n,
\]

and let \( z_i, w_i \) be complex variables with

\[
(2.2) \quad w_i = z_i q^{-u_i}, \quad i = 0, 1, \ldots, n.
\]

Then

\[
L := \prod_{1 \leq i < j \leq n} (t_j - t_i) \, dq_1 \cdots dq_n
\]

\[
(2.3) \quad = \left(-1\right)^{\sigma} q^{\tau} \frac{\Gamma_q(s_0)\Gamma_q(s_1)\cdots\Gamma_q(s_n)}{\Gamma_q(s_0 + s_1 + \cdots + s_n)} \prod_{0 \leq i < j \leq n} \frac{z_i q^k z_j}{z_j q^k z_i},
\]

where

\[
(2.4) \quad \sigma = \sum_{i=1}^n i s_i, \quad \tau = \sum_{i=1}^n i \left(\frac{s_i}{2}\right).
\]

**Remark 1.** Suppose that all \( z_i \) are nonzero and all \( u_i \) are zero. Then the integral formula in Theorem 1 can be written in the form

\[
\prod_{1 \leq i < j \leq n} (t_j - t_i) \, dq_1 \cdots dq_n
\]

\[
(2.5) \quad = \frac{\Gamma_q(s_0)\Gamma_q(s_1)\cdots\Gamma_q(s_n)}{\Gamma_q(s_0 + s_1 + \cdots + s_n)} \prod_{0 \leq i < j \leq n} \frac{z_i q^k z_j}{z_j q^k z_i},
\]
Since (2.5) is valid for all positive integers \( s_i \) by Theorem 1, it follows by analytic
continuation (cf. [3, p. 115]) that it holds for all complex \( s_i \) with
\[
\text{Re} (s_i) > \max_{0 \leq j \leq n} \frac{\log |z_j/z_i|}{\log q}, \quad i = 0, 1, 2, \ldots , n.
\]
If \( n = 1 \), (2.5) reduces to the Andrews–Askey \( q \)-integral [4, (2.2)].

**Remark 2.** From (2.5) and [9, Thm. 2.2], it may be deduced that the constant
term of the Laurent polynomial
\[
P(z_1, \ldots , z_n) := \int_0^1 \cdots \int_0^1 \prod_{0 \leq i, j \leq n} (qtz_j/z_i)_{s_i-1}
\]
\[
\cdot \prod_{1 \leq i < j \leq n} (t_j - t_i z_i/z_j) \, dt_1 \cdots dt_n
\]
equals
\[
(2.6) \prod_{i=1}^n (1 - q)/(1 - q^{s_i+s_{i+1}+\cdots+s_n}).
\]
It would be interesting to find a proof independent of [9].

**Proof of Theorem 1.** Assume that each \( z_i \) is an integral power of \( q \) and that the
sequence \( w_0, w_1, w_2, \ldots, w_n \) is monotone. It suffices to prove (2.3) under these assump-
tions, since both sides of (2.3) are polynomials in \( z_0, \ldots , z_n \).

Consider any one of the rightmost factors in (2.3), say
\[
(2.7) z_{\alpha} - q^\gamma z_{\beta},
\]
with
\[
(2.8) 0 \leq \alpha < \beta \leq n, \quad 1 - \gamma \leq \gamma \leq s_{\alpha} - 1.
\]
We will show that \( z_{\alpha} - q^\gamma z_{\beta} \) is also a factor of \( L \) by showing that \( L \) vanishes under
the assumption
\[
(2.9) z_{\alpha} = q^\gamma z_{\beta}.
\]
The \( q \)-integral \( L \) is a series by definition, and it suffices to show that each summand
in this series vanishes. This will be accomplished if we can show
\[
(2.10) \prod_{k=1}^{s_{\alpha}-1} (z_{\alpha} - q^kt) \prod_{m=1}^{s_{\beta}-1} (z_{\beta} - q^mt) = 0 \quad \text{for all } t \in S,
\]
where \( S \) is the set of integral powers of \( q \) between \( w_\alpha \) and \( w_\beta \) including \( \max \{ w_\alpha, w_\beta \} \)
but not \( \min \{ w_\alpha, w_\beta \} \). Define
\[
(2.11) A = \{ z_{\alpha}q^{-k} : 1 \leq k \leq s_{\alpha} - 1 \}, \quad B = \{ z_{\beta}q^{-m} : 1 \leq m \leq s_{\beta} - 1 \}.
\]
Since \( z_{\alpha} = q^\gamma z_{\beta} \) by (2.9), there is no integral power of \( q \) lying strictly between the sets
\( A \) and \( B \) on the real axis. It is thus seen that \( A \cup B = S \), and (2.10) follows. We have
now proved that \( L \) is divisible by each of the linear factors in (2.7), and hence by the
polynomial
\[
(2.12) \prod_{0 \leq i < j \leq n} \prod_{k=1}^{s_i-1} (z_i - q^kz_j).
\]
By definition of \( L \), if we view \( L \) as a polynomial in \( z_0 \) with leading term \( C_nz_0^n \) (with
\( C_n \) independent of \( z_0 \)), then
\[
(2.13) \nu = n(s_0 - 1) + (s_1 + \cdots + s_n).
\]
Viewing (2.12) as a polynomial in $z_0$, we see that it also has degree $\nu$. Thus it remains to prove that

$$C_n = (-1)^\nu q^{\sum_i \Gamma_q(s_i)} \prod_{1 \leq i < j \leq n} (z_i - q^k z_j).$$

(2.14)

First consider the case $n = 1$. Then $C_1$ is the coefficient of $z_0^{s_0 + s_1 - 1}$ in

$$\int_{t = w_0}^{w_1} \prod_{i = 1}^{s_0 - 1} (z_0 - q^k t) \int_{m = 1}^{s_1 - 1} (z_1 - q^m t) \, dq t,$$

(2.15)

so $C_1$ is the coefficient of $z_0^{s_0 + s_1 - 1}$ in

$$-\prod_{m = 1}^{s_1 - 1} (-q^m) \int_{t = w_1}^{w_0} t^{s_1 - 1} z_0^{s_0 - 1} (qt/z_0) \, dq t.$$

(2.16)

Replace $t$ by $z_0 t$ to see that $C$ is the constant term in the expansion in $z_0$ of

$$(-1)^{s_1} q^{\left(\begin{array}{c} s_1 \\ 2 \end{array}\right)} \int_{w_1/z_0}^{q^{-w_0}} t^{s_1 - 1} (qt/z_0) \, dq t.$$

(2.17)

The constant term in (2.17) is unchanged if the lower limit of $q$-integration is replaced by 0. It is further unchanged if the upper limit of $q$-integration is replaced by 1, since

$$\left(\begin{array}{c} q^m \\ 2 \end{array}\right) = 0 \quad \text{for} \quad t = q^{-t} \quad (i = 1, 2, \ldots, s_0 - 1).$$

(2.18)

It now follows from (1.7) that (2.14) holds for $n = 1$, so the proof of Theorem 1 is complete in the case $n = 1$.

Suppose now that $n > 1$ and that Theorem 1 holds with $(n-1)$ in place of $n$. Directly from (2.3), we see that $C_n$ is the coefficient of $z_0^{s_0 + \cdots + s_n - 1}$ in

$$\int_{t_n = w_{n-1}}^{w_n} \cdots \int_{t_2 = w_1}^{w_2} \prod_{i = 1}^{s_0} \prod_{j = 2}^{s_1} \prod_{k = 1}^{s_2} (z_i - q^k t_j) \cdot \prod_{2 \leq i < j \leq n} (t_j - t_i) \cdot (-1)^{s_1 + \cdots + s_n} q^{\left(\begin{array}{c} s_2 + \cdots + s_n \\ 2 \end{array}\right)} q^{\left(\begin{array}{c} s_1 + \cdots + s_n \\ 2 \end{array}\right)} \int_{t = w_1}^{w_0} t^{s_1 + \cdots + s_n - 1} z_0^{s_0 - 1} \, dq t dq t_2 \cdots dq t_n.$$

(2.19)

The inner integral on $t$ in (2.19) may be replaced by

$$z_0^{(s_0 + \cdots + s_n) - 1} \int_{w_1/z_0}^{q^{-w_0}} t^{(s_1 + \cdots + s_n) - 1} (qt) \, dq t,$$

(2.20)

and just as with (2.17), the desired coefficient is unchanged if we further replace the lower and upper limits of $q$-integration in (2.20) by zero and 1, respectively. Thus by (1.7), $C_n$ is the constant term of the polynomial in $z_0$ obtained from (2.19) by replacing the inner integral on $t$ by

$$\Gamma_q(s_0) \Gamma_q(s_1 + \cdots + s_n) \prod_{k = 1}^{s_2} (z_0 - q^k t) \, dq t dq t_2 \cdots dq t_n.$$

(2.21)

By induction on $n$, the proof of Theorem 1 is complete.

3. Proof of the $q$-Selberg integral formula. In this section we apply Theorem 1 to give a short proof of the $q$-Selberg integral formula (1.8). The result is true for $n = 1$ by (1.7), so let $n > 1$. We may assume that $a$ and $b$ are positive integers, as the result can be extended by analytic continuation to hold whenever Re $(a)$, Re $(b) > 0$. 


Given polynomials

\[(3.1)\quad E(t) = \prod_{i=1}^{n} (t - e_i), \quad H(t) = \prod_{i=1}^{n-1} (t - h_i)\]

with

\[(3.2)\quad 0 \leq e_1 \leq h_1 \leq e_2 \leq h_2 \leq \cdots \leq h_{n-1} \leq e_n \leq 1,\]

use for brevity the symbolic notation

\[(3.3)\quad \int_{E \in D_n} \{ \} \, dqE := \int_{e_n=0}^{1} \cdots \int_{e_2=0}^{e_1} \int_{e_1=0}^{e_2} \{ \} \cdot \prod_{1 \leq i < j \leq n} (e_i - e_j) \, dq_{e_1} \, dq_{e_2} \cdots dq_{e_n}\]

and

\[(3.4)\quad \int_{H \in D_{n-1}(E)} \{ \} \, dqH := \int_{h_{n-1}=e_{n-1}}^{e_n} \cdots \int_{h_2=e_2}^{h_1=e_1} \int_{h_1=0}^{h_2} \{ \} \cdot \prod_{1 \leq i < j \leq n-1} (h_i - h_j) \, dq_{h_1} \, dq_{h_2} \cdots dq_{h_{n-1}}.\]

Note that

\[(3.5)\quad \int_{E \in D_n} \int_{H \in D_{n-1}(E)} = \int_{H \in D_{n-1}} \int_{E \in D_n(V)},\]

where

\[(3.6)\quad V(t) = \prod_{i=0}^{n} (t - v_i) \quad \text{with} \quad v_0 = 0, \quad v_n = 1, \quad v_i = qh_i \quad (1 \leq i \leq n - 1).\]

Define

\[(3.7)\quad I_n(a, b, c) := \int_{E \in D_n} \int_{H \in D_{n-1}(E)} \prod_{i=1}^{n} e_i^{a-1}(qe_i)_{b-1} \cdot \prod_{i=1}^{n-1} \prod_{j=1}^{c-1} \prod_{k=1}^{i-1} (q^{c-1}e_i - q^{k+c-1}e_j) \, dqH \, dqE.\]

If we replace \(n\) by \(n-1\) in Theorem 1 and then further take \(t_i = h_i, s_i = c, u_i = c - 1, w_i = e_{i+1}, z_i = q^{c-1}e_{i+1}\), then Theorem 1 yields

\[(3.8)\quad I_{n-1}(a, b, c) = (-1)^{\binom{n-1}{2}} + b \binom{c}{2} \binom{c}{2} \frac{\Gamma_q(c)^n}{\Gamma_q(cn)} \cdot \prod_{1 \leq i < j \leq n} (q^{c-1}e_i - q^{k+c-1}e_j).\]

Thus, by definition of \(S_n(a, b, c)\) and \(I_n(a, b, c)\),

\[(3.9)\quad I_n(a, b, c) = (-1)^{\binom{n-1}{2}} + b \binom{c}{2} \binom{c}{2} \frac{\Gamma_q(c)^n}{\Gamma_q(cn)} S_n(a, b, c).\]
By (3.5) and (3.6), interchange of integration in (3.7) yields

\[
I_n(a, b, c) = \int_{H \in D_{n-1}} \int_{E \in D_{n}(V)} (-1)^{n(a-1)} q^{-n(\frac{c}{2})} \prod_{j=1}^{n} \prod_{k=1}^{a-1} (0 - q^{k} e_j) \\
\cdot \prod_{j=1}^{b-1} \prod_{k=1}^{b-1} (1 - q^{k} e_j) \\
\cdot 2^{c(n-1)} \prod_{i=1}^{n-1} \prod_{j=1}^{n} \prod_{k=1}^{c-1} (v_i - q^{k} e_j) \, d_q E \, d_q H.
\]

(3.10)

Apply Theorem 1 with \( t_i = e_i, \ s_0 = a, \ s_n = b, \ s_i = c \ (1 \leq i \leq n-1), \ u_i = 0, \ w_i = v_i, \) and \( z_i = v_i \) to see that the inner integral on \( E \) equals

\[
(-1)^{\left(\frac{n-1}{2}\right) + c(n-1)} q^{\left(\frac{c}{2}\right) \left(\frac{c-1}{2}\right) + 2\left(\frac{c}{2}\right)}
\]

(3.11)

\[
\frac{\Gamma_q(a)\Gamma_q(b)\Gamma_q(c)^{n-1}}{\Gamma_q(a+b+(n-1)c)} \prod_{j=1}^{n-1} \prod_{k=1}^{b-1} (1 - q^{k} v_j) \\
\cdot \prod_{1 \leq i < j \leq n-1}^{c-1} (v_i - q^{k} v_j).
\]

Before integrating (3.11) on \( H \), make the change of variables \( h_i \to q^{c-1} h_i \) (so \( v_i \to q^{c} v_i \)). As a result,

\[
I_n(a, b, c) = (-1)^{\left(\frac{n-1}{2}\right) + c(n-1)} q^{\left(\frac{c}{2}\right) \left(\frac{c-1}{2}\right) + 2\left(\frac{c}{2}\right) + (c-1)\left(\frac{c-1}{2}\right) + c(a+c-1)(n-1)}
\]

(3.12)

\[
\cdot \frac{\Gamma_q(a)\Gamma_q(b)\Gamma_q(c)^{n-1}}{\Gamma_q(a+b+(n-1)c)} S_{n-1}(a+c, b+c, c).
\]

Comparison of (3.9) and (3.12) yields

\[
S_n(a, b, c) = q^{ac(n-1)+c^2\left(\frac{n-1}{2}\right)} \frac{\Gamma_q(a)\Gamma_q(b)\Gamma_q(c)^{n-1}}{\Gamma_q(a+b+(n-1)c)\Gamma_q(c)} \cdot S_{n-1}(a+c, b+c, c)
\]

(3.13)

and the result follows by induction on \( n \). \( \square \)

4. Extension of the \( q \)-Selberg integral. Let \( S_{n,m}(a, b, c) \) denote the extension of the \( q \)-Selberg integral \( S_n(a, b, c) \) obtained by inserting the factor \( t_1 t_2 \cdots t_m \) in the integrand in (1.8), where \( 0 \leq m \leq n \). In Theorem 2 below, we evaluate \( S_{n,m}(a, b, c) \). It is not difficult to show that Theorem 2 is equivalent to the case \( l = 0 \) of [14, Thm. 2]; see [14, eqns. (4.17), (4.19)].

**Theorem 2.** For positive integers \( n, c \) and \( \text{Re} (a), \text{Re} (b) > 0 \),

\[
S_{n,m}(a, b, c) = \frac{S_n(a, b, c) T_{n,m}(a, b, c)}{\binom{n}{m}},
\]

where

\[
T_{n,m}(a, b, c) := q^{c(n-1)} \prod_{i=n-m}^{n-1} \frac{(1 - q^{a+c+i})(1 - q^{c+i})}{(1 - q^{a+b+c(n-1+i)})(1 - q^{cn-c-i})}.
\]

(4.1)

(4.2)
Proof. We proceed as in the proof in § 3, with the following modifications. Let \( u \) be an indeterminate and let \( S_n(a, b, c, u) \) be the extension of the \( q \)-Selberg integral \( S_n(a, b, c) \) obtained by inserting the factor \( \prod_{i=1}^n (u - t_i) \) in the integrand of (1.8). We must show that

\[
S_n(a, b, c, u) = \sum_{m=0}^n (-1)^m T_{n,m}(a, b, c) u^{n-m}.
\]

Let \( I_n(a, b, c, u) \) be the extension of \( I_n(a, b, c) \) obtained by inserting the factor \( q^{c(n-1)} H(u/q) \) in the integrand in (3.7). By Lagrange interpolation,

\[
q^{c(n-1)} H\left(\frac{u}{q}\right) = \sum_{r=1}^n q^{c(n-1)} H\left(\frac{e_r}{q}\right) \prod_{i \neq r} \frac{u - e_i}{e_r - e_i},
\]

for distinct \( e_i \). Thus, from (3.7),

\[
I_n(a, b, c, u) = \int_{E \in D_n} \sum_{r=1}^n \prod_{i \neq r} \frac{u - e_i}{e_r - e_i} e_i^{a-1}(q e_i) b_{-1} \\
\cdot \int_{H \in D_{n-1}(E)} \prod_{i=1}^n \prod_{j=1}^{n-1} \prod_{k=1}^{n-1} (q^{c-1} e_i - q^k h_j) d_u H d_H e_i,
\]

where \( \delta(i, r) = 1 \) if \( i = r \) and \( \delta(i, r) = 0 \) if \( i \neq r \). If for each fixed \( r \) we replace \( n \) by \( n-1 \) in Theorem 1, and then further take \( t_i = h_i, s_i = c + \delta(i, r), u_i = c-1, w_i = e_i+1, \) and \( z_i = q^{c-1} e_{i+1} \), then Theorem 1 shows that the inner integral on \( H \) in (4.5) equals

\[
\text{RHS}(3.8) \quad q^{c(n-1)(2c-1)} \left(1 - q^c\right) \prod_{i \neq r} (q^c e_r - e_i),
\]

where \( \text{RHS}(3.8) \) denotes the right-hand side of (3.8). Thus

\[
I_n(a, b, c, u) = q^{c(n-1)(2c-1)} \frac{1 - q^c}{1 - q^c n} \int_{E \in D_n} \text{RHS}(3.8) \\
\cdot \prod_{i=1}^n e_i^{a-1}(q e_i) b_{-1} \sum_{r=1}^n \prod_{i \neq r} \frac{u - e_i}{e_r - e_i} (q^c e_r - e_i) d_u H d_H e_i.
\]

Given a polynomial \( F(u) \), let \( F^*(u) \) denote its \( q^{-c} \)-derivative [11, p. 22], namely

\[
F^*(u) = \frac{F(u) - F(q^{-c}u)}{u - q^{-c}u}.
\]

Since

\[
E^*(e_r) = \prod_{i \neq r} (q^{-c} e_r - e_i),
\]

the inner sum on \( r \) in (4.7) equals \( E^*(u) \). Thus

\[
I_n(a, b, c, u) = \text{RHS}(3.9) q^{c(n-1)(2c-1)} \frac{1 - q^c}{1 - q^c n} S_n^*(a, b, c, u) \frac{S_n(a, b, c)}{S_n(a, b, c)}.
\]

After interchanging the order of integration, we obtain

\[
I_n(a, b, c, u) = \text{RHS}(3.12) q^{c(n-1)(2c-1)} S_{n-1}(a + c, b + c, c, uq^{-c}) \frac{S_{n-1}(a + c, b + c, c)}{S_{n-1}(a + c, b + c, c)}.
\]

Comparing (4.10) and (4.11), we arrive at the “differential equation”

\[
S_n^*(a, b, c, u) = 1 - q^c S_{n-1}(a + c, b + c, c, uq^{-c}) \frac{1 - q^c}{1 - q^c} S_{n-1}(a + c, b + c, c).
\]
By induction on $n$, (4.3) furnishes a solution to (4.12). Moreover, (4.3) is valid for $u = 0$, by (1.8) with $a + 1$ in place of $a$. Hence (4.3) is proved.


REFERENCES