Math 140B Final 155 points June 12, 2014

Directions: Justify all answers. If you appeal to a theorem, show that the hypotheses of that theorem are satisfied.

(1) If F' is Riemann integrable on $[a, b]$, prove that $\int_a^b F'(x)dx = F(b) - F(a)$. (25 pts)

SOLUTION: See Theorem 6.21.

(2) Find the limit of $(x + x^{2/3})^{1/3} - x^{1/3}$ as $x \to \infty$. (20 pts)

SOLUTION: Let f denote the cube root function, and let $y = x + x^{2/3}$. By the Mean Value Theorem, $f(y) - f(x) = f'(c)(y - x) = \frac{1}{3}(x/c)^{2/3}$, with c between x and y. Since $x/y < x/c < x/x$ and $x/y \to 1$ as $x \to \infty$, we have $x/c \to 1$ as $x \to \infty$, so the desired limit is $1/3$.

(3) For the function $f(x) = e^x \cos(x)$, write down the Taylor polynomial $P(x)$ of degree 3 expanded about $x = 0$. Then show that for $x \in (0, 1/2)$, $|f(x) - P(x)| < .02$. (25 pts)

SOLUTION: $f(x) = e^x \cos(x)$, $f'(x) = e^x(\cos(x) - \sin(x))$, $f''(x) = -2e^x \sin(x)$, $f'''(x) = -2e^x(\cos(x) + \sin(x))$, and $f''''(x) = -4e^x \cos(x)$. Thus $P(x) = 1 + x - \frac{1}{3}x^3$, and by Taylor's Theorem, there exists c between 0 and x such that $|f(x) - P(x)| \leq \frac{4e^c \cos(c) x^4}{4!} \leq \sqrt{e}/96 < .02$.

(4) Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (0, 1)$, where the c_n are real. True or False: If the limit of $f(x)$ exists as $x \to 1$, then $\sum_{n=0}^{\infty} c_n$ must converge. (15 pts)

SOLUTION: False–choose $c_n = (-1)^n$, so $f(x) = 1/(1 + x)$.

(5) Let $I = (-1, 1)$. Starting with the identity $\sum_{n=0}^{\infty} x^n = 1/(1 - x)$ for $x \in I$, prove that (A) $\sum_{n=1}^{\infty} n x^{n-1} = 1/(1-x)^2$, for $x \in I$; (B) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \ln(1+x)$, for $x \in I$; (C) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$. Give precise statements of all major theorems that you use in your proofs. (35 pts)

SOLUTION: (A) The radius of convergence is 1. Let $0 < c < 1$. The series converges uniformly on $[-c, c]$ by Theorem 8.1. Thus term by term differentiation is valid on $[-c, c]$ by Theorem 7.17 (or the simplified version of that theorem involving continuous derivatives). (B) Replace x by $-x$ in the geometric series and integrate term by term from 0 to x. This is valid by Theorem 7.16. (C) Let $x \to 1$ in (B) and apply Abel's theorem. Abel's theorem may be applied because the left side of (C) converges by the alternating series test.

(6) Give an example of a sequence of functions $\{f_n\}$ on $(0, 1)$ satisfying $0 \leq f_n \leq f_{n+1} \leq 1$ for all n, such that the sequence converges pointwise but not uniformly to a continuous function on $(0, 1)$. (15 pts)

SOLUTION: Take $f_n(x)$ to be 1 or 0 according as $0 < x \leq \frac{n}{n+1}$ or $\frac{n}{n+1} < x < 1$. Then $f_n \to f = 1$ pointwise, but not uniformly on $(0, 1)$, since $|f_n(\frac{n+1}{n+2}) - f(\frac{n+1}{n+2})| = |0 - 1| = 1$ for all n.

(7) Using partitions, show that the sum of two Riemann-Stieltjes integrable functions on $[0, 1]$ is again a Riemann-Stieltjes integrable function on $[0, 1]$. (20 pts)

SOLUTION: See Theorem 6.12(a).