Math 140B Test 1 100 points April 25, 2014

Directions: Justify all answers. No calculators. If you appeal to a theorem, show that the hypotheses of that theorem are satisfied. As usual, $\mathbb{R}$ and $\mathbb{C}$ denote the reals and the complexes, respectively. Each problem is worth 20 points.

(1) Let X be a metric space disconnected by subsets A and B. Prove that X fails to be arcwise connected. *Hint*: Choose $p \in A$ and $q \in B$.

SOLUTION: Suppose for the purpose of contradiction that X were arcwise connected. Then there would be a continuous function $f : [0,1] \to X$ with $f(0) = p \in A$ and $f(1) = q \in B$. Let u be the sup of the set $\{t : f([0,t]) \subset A\}$. Then by continuity of f, either $f(u) \in A$ or $f(u)$ is a limit point of A. In either case, $f(u) \in A$ because A is closed. Similarly, $f(u) \in B$. Thus A and B are not disjoint, a contradiction.

(2) Define $g(x) = \cos(x) - 1 + \frac{1}{2}x^2$ for $x \in (0,1)$. Prove that for all $x \in (0,1)$, we have $0 < g(x) < .05$. *Hint*: Taylor.

SOLUTION: By Taylor's theorem for $f(x) = \cos(x)$ with $n = 4$ and $\alpha = 0$,

$$g(x) = \frac{f^{(4)}(c)}{4!}x^4 = \frac{\cos(c)}{4!}x^4$$

for some $c \in (0,x)$. Since $0 < \cos(c) < 1$ and $0 < x < 1$, we have

$$0 < g(x) < \frac{1}{4!} < .05.$$

(3) Describe the set of all points $z \in \mathbb{C}$ for which the series $\sum_{n=1}^{\infty} z^n/n$ converges.

SOLUTION: By comparison with the geometric series, the given series converges absolutely for all z strictly inside the unit disk centered at the origin. Hence the series converges inside this disk, so the radius of convergence is at least 1. But the series diverges when $z = 1$, so the radius of convergence cannot be larger than 1. Hence the radius of convergence is 1. For each z on the unit circle except for $z = 1$, the series converges by Theorem 3.42 with $a_n = z^n$ and $b_n = 1/n$.

(4) Define $f(x) = \exp(-1/x^2)$ for nonzero x, and define $f(0) = 0$. Find $f'(0)$. *Hint*: If positive h tends to zero, then $u = 1/h$ tends to infinity.

SOLUTION: We will show that $f'(0) = 0$. By definition of the derivative at 0, we need to show that 0 equals the limit of $\frac{\exp(-1/h^2)}{h}$ as $h \to 0$. It suffices to show that this limit is 0 as $h \to 0+$, since the same argument will also show that the limit is 0 as $h \to 0-$. Let $u = 1/h$. Then the limit in question equals the limit as $u \to \infty$ of $u \exp(-u^2)$. This is the limit of $\frac{u}{\exp(u^2)}$. Since the denominator approaches ∞ as $u \to \infty$, L'Hopital's rule applies, and it shows the limit is 0.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ have a positive second derivative everywhere on $\mathbb{R}$. Given also that $f(0) = 0$, show that for all $x \geq 0$, we have

$$xf'(x) \geq f(x) \geq xf'(0).$$

SOLUTION: The assertion is true when $x = 0$, so assume that $x > 0$. Then $\frac{f(x)}{x} = f'(c)$ for some $c \in (0, x)$, by the MVT. As $f'' > 0$, we know that f' is increasing, again by the MVT. Thus $f'(x) > f'(c) > f'(0)$, which yields the desired result upon multiplication by x.