Math 140B   Test 2   100 points   May 23, 2014

Directions: Justify all answers. No calculators. If you appeal to a theorem, show that the hypotheses of that theorem are satisfied. The notation $[x]$ denotes the greatest integer $\leq x$. Each problem is worth 25 points.

(1) Evaluate $\int_0^2 f d\alpha$
(A) when $f(x) = e^x,\ \alpha(x) = e^x$;
(B) when $f(x) = e^x,\ \alpha(x) = [x]$.

SOLUTION: (A) This equals the integral from $x = 0$ to $x = 2$ of e^{2x}, which is $(e^4 - 1)/2$.
(B) Since α jumps by 1 at $x = 1$ and at $x = 2$, the integral equals $f(1) + f(2) = e + e^2$.

(2) True or False: The integral $\int_{-1}^1 f d\alpha$ exists
(A) when $f(x) = \sin(1/x),\ \alpha(x) = [x]$;
(B) when $f(x) = \sin(1/x),\ \alpha(x) = \sin(x)$.
Justify each True or False answer, using theorems if necessary.

SOLUTION: (A) False. There is a subinterval of the partition which contains the point $x = 0$ for which $M_i = 1$, $m_i = -1$, and $\Delta(\alpha_i) = 1$. Thus the corresponding upper sum U differs from the lower sum L by at least 2, so the integral does not exist.
(B) True, since α is continuous at the only discontinuity of the bounded function $f(x)$, namely the discontinuity at $x = 0$.

(3)

Let $f(x) = \sum_{k=1}^{\infty} \frac{1}{1 + xk^2}$ for $x \in (0, 1)$. Fix $c \in (0, 1)$.
(A) Show that this series converges uniformly on $(c, 1)$.
(B) Does the series converge uniformly on $(0, 1)$? Justify.
(C) Is $f(x)$ continuous on $(0, 1)$? Justify.

SOLUTION: (A) The k-th term is less than $1/(ck^2)$, so the series converges uniformly by the Weierstrass test.
(B) No. The series does not converge uniformly on $(0, 1)$, because for any large $M < N$, the sum from $k = M$ to $k = N$ is not uniformly small on

$(0, 1)$. For example, if $x = 1/M^2$, then the term for $k = M$ already fails to be small, since it equals $1/2$.
(C) Yes. Note that $f(x)$ is continuous on $(c, 1)$ by part (A). Since c can be chosen arbitrarily close to 0, $f(x)$ is continuous at every point in $(0, 1)$.

(4) Let $f_n(x) \to f(x)$ uniformly on $(0, 1)$, where each $f_n(x)$ is continuous on $(0, 1)$. Prove that $f(x)$ is continuous on $(0, 1)$.
Hint: Let $\epsilon > 0$ and fix $x \in (0, 1)$. Fix N such that $|f_N(u) - f(u)| < \epsilon$ for all $u \in (0, 1)$. Show that $f(t) \to f(x)$ as $t \to x$.

SOLUTION: Take $u = t$ and $u = x$ in the inequality given in the Hint. Thus

$$|f_N(x) - f(x)| < \epsilon, \quad |f_N(t) - f(t)| < \epsilon.$$

Since f_N is continuous, there is a δ for which $|t - x| < \delta$ implies

$$|f_N(t) - f_N(x)| < \epsilon.$$

Combining these three inequalities, we obtain $|f(t) - f(x)| < 3\epsilon$ when $|t - x| < \delta$.