Announcements

- If you cannot see your Midterm result on TED, make sure you are choosing the ‘All’ option in the Grade Center.
- Homework 3 due today.
Invariance

Game (Matching Pennies)

Both players simultaneously choose either ‘heads’ or ‘tails’. If the two choices are the same, then Player I wins 1. If they are different, Player II wins 1.

The payoff matrix for this game is:

\[
\begin{pmatrix}
H & T \\
H & (+1 & -1) \\
T & (-1 & +1)
\end{pmatrix}
\]

Notice that if we swap ‘H’ and ‘T’ in this matrix, the payoff matrix remains the same:

\[
\begin{pmatrix}
T & H \\
T & (+1 & -1) \\
H & (-1 & +1)
\end{pmatrix}
\]
Reordering the rows and columns of the payoff matrix does not change the rules of the game, but usually this will change the matrix itself.

**Def.** A matrix game is invariant under some reordering of rows and columns if the payoff matrix is the same after we perform the reordering.

For example, the *Matching Pennies* game was invariant under the reordering: ‘H’ → ‘T’, ‘T’ → ‘H’.

Invariance

Theorem

Let \( A \) be a matrix game that is invariant under some reordering.

- If \( x, y \) are two rows, where \( x \rightarrow y \) under the reordering, then there is an optimal strategy for Player I with \( p(x) = p(y) \).
- If \( x, y \) are two columns, where \( x \rightarrow y \) under the reordering, then there is an optimal strategy for Player II with \( q(x) = q(y) \).

For example, use this to solve *Matching Pennies*:

\[
\begin{pmatrix}
H & T \\
H & \begin{pmatrix} +1 & -1 \\
T & \begin{pmatrix} -1 & +1 \\
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\]
Consider the following game:

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 0 & -1 & 1 \\
1 & -1 & 0 & -1 \\
2 & 1 & -1 & 0 \\
\end{array}
\]

Solve this game using Invariance.
**Note About Invariance**

**Note:** In the textbook, the section on Invariance (Part II, 3.6) relies heavily on group theory (in particular, group actions). Since MA100 is not a prerequisite, I will not assume any knowledge of this. For section 3.6, all that you need to know is what is contained in these slides.
For this section we will investigate what happens if a player knows what (mixed) strategy their opponent is going to use.

First we establish some notation. Remember that $X$ and $Y$ are the sets of pure strategies for Player I and Player II respectively.

Let $X^*$ and $Y^*$ be the set of all mixed strategies for Player I and Player II respectively.
Best Responses

- Assume that Player I knows that Player II will use strategy $q \in Y^*$. Then Player I should choose a strategy $p$ that maximizes their average winnings.
- **Def.** Given a (mixed) strategy $q$ for Player II, the best response strategy for Player I is the strategy $p$ that maximizes $p^T A q$.
- **Def.** Given a (mixed) strategy $p$ for Player I, the best response strategy for Player II is the strategy $q$ that minimizes $p^T A q$.

**Question**

Given Player II’s strategy is $q = [0.5 \ 0.2 \ 0.3]^T$, find the best response strategy for Player I, for the game below.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{pmatrix}$$
As in the previous slide, assume that Player II must announce their strategy $q$ before Player I chooses their strategy.

For a finite game, the best response strategy can always be taken to be a pure strategy: Player I will choose whichever row gives them the biggest average payoff (this is a pure strategy).

Player II wants to choose $q$ so that Player I’s best response payoff is as small as possible. Let $V$ be this minimum payoff. It is called the upper value of the game.
Best Response: Upper Value

- From the previous slide, $\bar{V}$ is the least amount that Player II will lose, on average, if they must announce their strategy before Player I chooses theirs.

- Written mathematically, Player II must solve:

  $$\bar{V} = \min_{q \in Y^*} \max_{1 \leq i \leq m} \sum_{j=1}^{n} a_{ij}q_j = \min_{q \in Y^*} \max_{p \in X^*} p^TAq$$

- The strategy $q$ that attains this minimum is called the minimax strategy for Player II. It is the same as the optimal strategy we defined before.

- Since announcing their strategy first cannot be an advantage, we have $V \leq \bar{V}$.
• Similarly, $\overline{V}$ is the least amount that Player I will win, on average, if they must announce their strategy before Player II chooses theirs.

• Written mathematically, Player I must solve:

$$V = \max_{p \in X^*} \min_{1 \leq j \leq n} \sum_{i=1}^{m} p_i a_{ij} = \max_{p \in X^*} \min_{q \in Y^*} p^T A q$$

• $\underline{V}$ is the lower value of the game, and the $p$ attaining the max will be the same as the optimal strategy we found before.

• Since announcing their strategy first cannot be an advantage, we have $\underline{V} \leq V \leq \overline{V}$. 