Existence of a Strategic Equilibrium

Theorem (Nash, 1950)

Every bimatrix game has at least one mixed strategic equilibrium.

- This was proved by John Nash in 1950, whose life was depicted in the film *A Beautiful Mind*.
- For this reason, strategic equilibria are also called Nash equilibria.
- More generally, he showed that for any finite number of players, each with a finite number of pure strategies, there is at least one (mixed) strategic equilibrium.
Cournot Duopoly

Company A and B produce identical bottles of water. The cost of producing one bottle is \( c \), and the price that consumers are willing to pay is \( (a - Q)^+ \), where \( a \) is a fixed constant and \( Q \) is the total number of bottles produced by each firm. Both companies choose the amount they produce simultaneously, and all bottles are then sold at the value given above.

How much should each firm produce?

This question was studied by Antoine Cournot in 1838.

(The notation \( x^+ \) means \( x \) when \( x \geq 0 \), and 0 if \( x < 0 \)).
Company A and B produce bottled water. The cost of producing one bottle is $c$, and the price that consumers are willing to pay is $(a - Q)^+$, where $a$ is a fixed constant and $Q$ is the total number of bottles produced by each firm.

Let $q_1$ be the amount produced by Company A, and $q_2$ the amount produced by Company B. Hence $Q = q_1 + q_2$.

The payoff for Company A is then

$$u_1(q_1, q_2) = q_1(a - q_1 - q_2)^+ - cq_1$$

and for Company B:

$$u_2(q_1, q_2) = q_2(a - q_1 - q_2)^+ - cq_2$$
Cournot Duopoly: Monopoly Situation

- As a first step, let us analyze this game where there is just one company.
- The payoff is
  \[ u_1(q_1, 0) = q_1(a - q_1)^+ - cq_1 \]
- To maximize this, we set the derivative equal to zero:
  \[
  \frac{\partial u_1}{\partial q_1}(q_1, 0) = \begin{cases} 
  a - 2q_1 - c, & a - q_1 \geq 0 \\
  -c, & a - q_1 < 0
  \end{cases}
  \]
- This partial derivative is equal to zero only when
  \[ q_1 = \frac{a - c}{2} \]
- In this case, the payoff is
  \[
  u_1\left(\frac{a - c}{2}, 0\right) = \frac{a - c}{2} \frac{a + c}{2} - \frac{ac - c^2}{2} = \frac{(a - c)^2}{4}
  \]
In summary, if there is just one firm:

- The firm will produce \((a - c)/2\).
- The profit is \((a - c)^2/4\).
- The cost, per bottle, to the consumer is

\[
(a - Q)^+ = \frac{a + c}{2}
\]
Now we return to the situation with two companies.

For a fixed $q_2$, we can find the values of $q_1$ where Company A has no incentive to change $q_1$. Again, we find when the derivative is zero:

$$\frac{\partial u_1}{\partial q_1}(q_1, q_2) = \begin{cases} a - 2q_1 - q_2 - c, & a - q_1 - q_2 \geq 0 \\ -c, & a - q_1 - q_2 < 0 \end{cases}$$

Solving, we get

$$q_1 = \frac{a - c - q_2}{2}$$

The interpretation is: for a fixed value of $q_2$, Company A’s best response is $q_1 = (a - c - q_2)/2$. 
We have seen that for a fixed $q_2$, Company A’s best response is
$$q_1 = \frac{a - c - q_2}{2}.$$  
By symmetry, for a fixed $q_2$, Company B’s best response is
$$q_2 = \frac{a - c - q_1}{2}.$$  
If we solve these simultaneously, we find a pair of strategies that are best responses to each other.
Since neither player has incentive to switch, this is a pure strategic equilibrium.
Solving, we find
$$q_1 = q_2 = \frac{a - c}{3}$$
So we have found a pure strategic equilibrium with

\[ q_1 = q_2 = \frac{a - c}{3} \]

The profits for each company are given by

\[ \frac{a - c}{3} \cdot \left( a - \frac{2(a - c)}{3} - c \right) = \frac{(a - c)^2}{9} \]

The cost, per bottle, to the consumer is given by

\[ \left( a - \frac{2(a - c)}{3} \right) = \frac{a + 2c}{3} \]
### Duopoly vs Monopoly Comparison

<table>
<thead>
<tr>
<th></th>
<th>Amount Produced</th>
<th>Price</th>
<th>Company A</th>
<th>Company B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Monopoly</strong></td>
<td>$\frac{a - c}{2}$</td>
<td>$\frac{a + c}{2}$</td>
<td>$\frac{(a - c)^2}{4}$</td>
<td>0</td>
</tr>
<tr>
<td><strong>Duopoly</strong></td>
<td>$\frac{2(a - c)}{3}$</td>
<td>$\frac{a + 2c}{3}$</td>
<td>$\frac{(a - c)^2}{9}$</td>
<td>$\frac{(a - c)^2}{9}$</td>
</tr>
</tbody>
</table>
We have found a pure strategic equilibrium for the Cournot Duopoly setup.

Could there be another equilibrium?

**Note:** There are uncountably many pure strategies (any nonnegative real). So a mixed strategy here is *any probability distribution on the nonnegative reals*. This is a huge space!

Luckily, a minor miracle occurs: we can iteratively remove dominated strategies, and all that will be left is the equilibrium we found above.
Models of Duopoly

Cournot Duopoly: Removing Dominated Strategies

- Recall that
  \[ u_1(q_1, q_2) = q_1(a - q_1 - q_1^+) - cq_1 \]

- A quick computation gives
  \[ \frac{\partial u_1}{\partial q_1}(q_1, q_2) < 0 \]
  iff \( q_1 > (a - c - q_2)/2 \).

- Notice that if \( q_1 > (a - c)/2 \), then this inequality is satisfied for all \( q_2 \). It follows that \( q_1 = (a - c)/2 \) dominates any \( q_1 > (a - c)/2 \).

- So we can restrict our attention to \( q_1 \leq (a - c)/2 \).

- By symmetry, we can also assume that \( q_2 \leq (a - c)/2 \).
By domination, we have found that $q_1 \leq (a - c)/2$ and $q_2 \leq (a - c)/2$.

A quick computation gives

$$\frac{\partial u_1}{\partial q_1}(q_1, q_2) > 0$$

iff $q_1 < (a - c - q_2)/2$.

Since $q_2 \leq (a - c)/2$, we have that

$$\frac{a - c}{4} \leq \frac{a - c - q_2}{2}$$

In particular, if $q_1 < (a - c)/4$, we get that $u_1$ is increasing, for all $q_2$.

So by domination we can assume $q_1 \geq (a - c)/4$.
Cournot Duopoly: Removing Dominated Strategies

In summary:

- Using domination, we removed all strategies where $q_1, q_2$ were larger than $(a - c)/2$.
- With these removed, by domination we could remove all strategies where $q_1, q_2$ were smaller than $(a - c)/4$.
- With these removed, we can remove even more strategies by domination...
- If we continue this process, one can verify that all strategies will be dominated except $q_1 = q_2 = (a - c)/3$. 
Question

Consider a modification of the Cournot setup, where the two companies have two different production costs, $c_1$ and $c_2$, find all pure strategic equilibria.