Announcements

- Midterm scores will be posted later today.
- The average was 85%.
- Keep in mind: there will be no definitions on the final (and people did very well on the definitions).
- Homework 6 will be posted this afternoon.
Entry Deterrence

Consider a monopoly, where the price demand relationship is given by

\[ P(Q) = \begin{cases} 
17 - Q & 0 \leq Q \leq 17 \\
0 & \text{otherwise}
\end{cases} \]

and where the cost of producing \( q_1 \) units is \( q_1 + 9 \).

What is the best choice of \( q_1 \)?

(Remark: Note that the production cost has a startup cost here, unlike our previous models.)
In this case, the payoff function is

\[ u_1(q_1) = q_1(17 - q_1) - (q_1 + 9) \]

The derivative is

\[ \frac{du_1}{q_1} = -2q_1 + 16, \]

so the maximum will be obtained at \( q_1 = 8 \).

So the payoff is \( u_1 = 8 \cdot 9 - 17 = 55 \).
After this firm has decided to set $q_1 = 8$, another firm considers entering the market.

They calculate that their payoff will be

$$u_2(q_2) = q_2(17 - 8 - q_2) - (q_2 + 9)$$

They optimize this,

$$\frac{du_2}{q_2} = -2q_2 + 8$$

So the max occurs when $q_2 = 4$, yielding a profit of $u_2(4) = 7$.

But now the first firm’s profits decrease. Firm I does not like this. If they could go back in time, can they prevent this?
Entry Deterrence

- Firm I is worried that a competitor will choose to produce $q_2$ units. The payoff for Firm II in this case is

$$u_2(q_1, q_2) = q_2(17 - q_1 - q_2) - (q_2 + 9)$$

- Firm I would like to choose $q_1$ in such a way that Firm II has no incentive to enter the market.

- For a constant $q_1$, let's find Firm II's response strategy $q_2$.

- Maximizing for $q_2$,

$$\frac{\partial u_2}{\partial q_2} = 17 - q_1 - 2q_2 - 1$$

So $q_2 = 8 - q_1/2$. 
Entry Deterrence

• The payoff function is

\[ u_2(q_1, q_2) = q_2(17 - q_1 - q_2) - (q_2 + 9) \]

• Firm II’s strategy will be \( q_2 = 8 - q_1/2 \), so

\[
\begin{align*}
    u_2(q_1, q_2) &= \left(8 - \frac{q_1}{2}\right)\left(17 - q_1 - 8 + \frac{q_1}{2}\right) - \left(8 - \frac{q_1}{2} + 9\right) \\
    &= \frac{q_1^2}{4} - 8q_1 + 55
\end{align*}
\]

• If we make this function 0, then Firm II will not enter the market.
• So \( \frac{q_1^2}{4} - 8q_1 + 55 = 0 \Rightarrow (q_1 - 10)(q_1 - 22) = 0 \)
• If \( q_1 = 10 \), then Firm I’s profits are \( u_1 = 51 \) (compare this to their initial profit of 55).
How does the situation differ if both firms choose production simultaneously (i.e., Cournot)?

Payoffs in this case are:

\[ u_1(q_1, q_2) = q_1(17 - q_1 - q_2) - (q_1 + 9) \]
\[ u_2(q_1, q_2) = q_2(17 - q_1 - q_2) - (q_2 + 9) \]

Setting derivatives to zero, we get

\[-2q_1 + 16 - q_2 = 0 = -2q_2 + 16 - q_1\]

So \( q_1 = q_2 = \frac{16}{3} \), yielding profits of \( 19\frac{4}{9} \) (compare this to 55 and 51).
Cooperative Games

- The general-sum games we have been discussing so far have been non-cooperative.
- As we have seen, a strategic equilibrium is self-enforcing: players have no incentive to switch from it, and so are inclined to choose these strategies without needing a binding agreement.
- However, if we allow binding agreements, players can do better.
- For example, the Prisoner’s dilemma:

\[
\begin{array}{cc}
C & D \\
C & (3, 3) & (0, 4) \\
D & (4, 0) & (1, 1) \\
\end{array}
\]

- If possible, the two prisoners will make a binding agreement to both Cooperate.
In the cooperative theory we allow players to make binding agreements.

We will consider two cases:

1. **Transferable Utility (TU):** players are allowed to make payments to each other when the game ends (called side-payments).
2. **Nontransferable Utility (NTU):** players are not allowed to make side-payments. The only payoff that occurs is from the game itself.
**Convex Sets**

**Def.** A subset $S$ of $\mathbb{R}^2$ is a **convex set** if every straight line joining two points in $S$ is completely contained in $S$. 

![Convex](image1.png) ![Not Convex](image2.png)

**Convex** **Not Convex**
Def. The convex hull of a set $T$ in $\mathbb{R}^2$ is the smallest convex set $S$ that contains $T$.

Given a finite set of points, we can easily find the convex hull, adding one point at a time.

Finding the convex hull is an important problem in computational geometry, and there are many algorithms (eg. the Graham Scan).
Example. Find the convex hull of $(0, 0), (0, 5), (2, 3), (3, 2), (5, 5)$. 

![Convex Hull Diagram](image-url)
**Example.** Find the convex hull of \((0, 0), (0, 5), (2, 3), (3, 2), (5, 5)\).
Convex Hull

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Feasible Sets

- Informally speaking, the *feasible set* of a game is the set of all possible payoffs that might occur as a result of the game (including side-payments, if allowed).

- **Def.** Consider a bimatrix game with payoff matrices $A, B$, where Player I has $m$ pure strategies and Player II has $n$ pure strategies. The NTU feasible set of this game is the convex hull of the $mn$ points $(a_{ij}, b_{ij})$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. 
Question

For the bimatrix game

\[
\begin{pmatrix}
(4, 3) & (0, 0) \\
(2, 2) & (1, 4)
\end{pmatrix}
\]

find the NTU feasible set.
Feasible Sets

- In a TU game, if \((x, y)\) is a possible payoff in a game, then so is \((x - s, y + s)\) for any constant \(s \in \mathbb{R}\). Here \(s\) is some sidepayment.

- **Def.** Consider a bimatrix game with payoff matrices \(A, B\). The **TU feasible set** of this game is the convex hull of the points \((a_{ij} + s, b_{ij} - s)\), where \(1 \leq i \leq m\) and \(1 \leq j \leq n\) and \(s\) is any real number.

- Note that for any point \((x, y) \in \mathbb{R}^2\), the set of points \((x - s, y + s)\) is just the line with slope \(-1\) that goes through \((x, y)\).

- In particular, the TU feasible set is just the NTU feasible set translated along the line with slope \(-1\).
Feasible Sets

Question
For the bimatrix game
\[
\begin{pmatrix}
4 & 3 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
2 & 2 \\
1 & 4
\end{pmatrix}
\]
find the TU feasible set.
Def. A payoff \((x, y)\) in the feasible set of a game is Pareto optimal if there is no other point in the feasible set \((x', y')\) with \(x' \geq x\) and \(y' \geq y\).

In the diagram of a feasible set, the points which are Pareto optimal will be the upper right boundary.

When the players come to a binding agreement, they will always choose a point that is Pareto optimal.
Solving TU Games

- The first thing to notice about solving a TU game is that both players will want to choose a point where the sum of the payoffs is maximized.

\[ \sigma = \max_i \max_j (a_{ij} + b_{ij}) \]

- What remains is to decide how to split \( \sigma \) between the two players.

- Both players will choose a threat strategy that they will follow if negotiations break down. Denote these by \( p \) and \( q \). The payoffs in this case are:

\[ (p^T A q, p^T B q) = (D_1, D_2) \]

- Player I will accept no less than \( D_1 \), Player II will accept no less than \( D_2 \). These correspond to the payoffs: \((D_1, \sigma - D_1)\) and \((\sigma - D_2, D_2)\).
Cooperative Games

Solving TU Games

- So Player I will accept no less than \((D_1, \sigma - D_1)\) and Player II will accept no less than \((\sigma - D_2, D_2)\), where \(D_1 = \mathbf{p}^T \mathbf{A} \mathbf{q}\) and \(D_2 = \mathbf{p}^T \mathbf{B} \mathbf{q}\).

- A natural compromise is the midpoint:

\[
\left(\frac{\sigma + D_1 - D_2}{2}, \frac{\sigma - D_1 + D_2}{2}\right)
\]

- So Player I wants to maximize

\[
D_1 - D_2 = \mathbf{p}^T (\mathbf{A} - \mathbf{B}) \mathbf{q}
\]

and Player II wants to minimize it.

- This is equivalent to playing the zero sum game \(\mathbf{A} - \mathbf{B}\).

- If \(\delta = \text{Val}(\mathbf{A} - \mathbf{B})\), then the TU solution is given by

\[
\left(\frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2}\right)
\]
Find the TU solution to the game

\[
\begin{pmatrix}
(0, 0) & (6, 2) & (-1, 2) \\
(4, -1) & (3, 6) & (5, 5)
\end{pmatrix}
\]

We know that the solution is given by

\[
\left( \frac{\sigma + \delta}{2}, \frac{\sigma - \delta}{2} \right),
\]

where

\[
\sigma = \max_i \max_j (a_{ij} + b_{ij})
\]

and

\[
\delta = \text{Val}(A - B)
\]