1. We have

\[ A^{-1} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix} \]

So \( 1^T A^{-1} 1 = (1/2 + 1/3 + 1 + 1/3) = 13/6 \). This is not zero, so we can use the nonsingular matrix theorem. The value of the game is \( V = (1^T A^{-1} 1)^{-1} = 6/13 \). The optimal strategies for each players are given by

\[ p^T = V 1^T A^{-1} = (3/13 \ 2/13 \ 6/13 \ 2/13) \]

and

\[ q^T = V 1 A^{-1} = (3/13 \ 2/13 \ 6/13 \ 2/13) \]

2. This is an upper triangular matrix, so let us try looking for an equalizing strategy. Let Player I’s strategy be \( p^T = (p_1 \ p_2 \ p_3 \ p_4) \) and let Player II’s strategy be \( q^T = (q_1 \ q_2 \ q_3 \ q_4) \). We get the following system of equations for Player I:

\[
\begin{align*}
    p_1 &= V \\
    -p_1 + p_2 &= V \\
    -2p_2 + p_3 &= V \\
    -p_1 + p_2 - p_3 + p_4 &= V \\
    p_1 + p_2 + p_3 + p_4 &= 1
\end{align*}
\]
Solving this, we get $V = 1/13$ and

$$p^T = \begin{pmatrix} 1/13 & 2/13 & 5/13 & 5/13 \end{pmatrix}$$

For Player II, we get the system

$$
\begin{align*}
q_1 - q_2 - q_4 &= V \\
q_2 - 2q_3 + q_4 &= V \\
q_3 - q_4 &= V \\
q_4 &= V \\
q_1 + q_2 + q_3 + q_4 &= 1
\end{align*}
$$

Solving this system, we get

$$q^T = \begin{pmatrix} 6/13 & 4/13 & 2/13 & 1/13 \end{pmatrix}$$

3. Denote the columns by A, B and C, and denote the rows by 1, 2 and 3. Now, swapping rows 1 and 3 and columns A and C leaves the payoff matrix unchanged. So, by Invariance, we can assume that the optimal strategies $p$ and $q$ satisfy $p(1) = p(3)$ and $q(A) = q(C)$.

We look for an equalizing strategy. For Player I, we get

$$
\begin{align*}
-4p(1) + p(2) + 2p(3) &= V \\
p(1) - 5p(2) + p(3) &= V \\
2p(1) + p(2) - 4p(3) &= V \\
p(1) + p(2) + p(3) &= 1
\end{align*}
$$

Now applying $p(1) = p(3)$, we get

$$
\begin{align*}
-2p(1) + p(2) &= V \\
2p(1) - 5p(2) &= V \\
2p(1) + p(2) &= 1
\end{align*}
$$

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Solving this system, we get $V = -\frac{1}{2}$ and
\[
\mathbf{p}^T = \begin{pmatrix} 3/8 & 1/4 & 3/8 \end{pmatrix}
\]
Since the matrix is symmetric, the system of equations for finding an equalizing strategy for Player II is the same as for Player I. So we also have
\[
\mathbf{q}^T = \begin{pmatrix} 3/8 & 1/4 & 3/8 \end{pmatrix}
\]
4. Answer 1: Note that the payoff matrix $A$ in this question satisfies $A(i, j) = i - j$. So we have
\[
A(i, j) = i - j = -(j - i) = -A(j, i)
\]
and so $A$ is skew-symmetric. Therefore the value of the game is zero.

Answer 2: We can note that the bottom right entry of the matrix (entry (1000, 1000)) is a saddle point, since $A(1000, 1000) = 0$, and 0 is the smallest number in the last row and the largest number in the last column. So $V = 0$.

Answer 3: Note that the last row dominates all other rows. Similarly, the last column dominates all other columns. All that remains is the bottom right entry, which is 0. So $V = 0$.

5. (a) Given this strategy for Player II, the possible payoffs are
\[
A\mathbf{q} = \begin{pmatrix} 2q + 1 \\ -4q + 5 \end{pmatrix}
\]
Player I will choose the first row if $2q + 1 > -4q + 5 \iff q > 2/3$, and the second row otherwise.

(b) Player II knows that this is what Player I will choose. So Player II will choose the value of $q$ that maximizes the outcome.
If $0 \leq q \leq 2/3$, then the payoff is $-4q + 5$. Along this interval, this function is minimized at $q = 2/3$. The payoff in this case is $7/3$.
If $2/3 < q \leq 1$, then the payoff is $2q + 1$, which is always greater than $7/3$ on this interval. So the minimum payoff that Player II can achieve is $7/3$, if they choose $q = 2/3$.  

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6. (a) Let \( C \) denote the matrix whose entries are all \( c \). So \( B = A + C \).

**Answer 1:** Let \( p, q \) be strategies chosen by players I and II during a game of \( B \). Then the payoff is

\[
p^T B q = p^T (A + C) q = p^T A q + c
\]

where the last equality follows from the fact that the entries of \( p \) and \( q \) sum to 1.

In particular, playing the game \( B \) is exactly the same as playing the game \( A \), and then adding \( c \) to the payoff of \( A \). It is clear that the optimal strategy for both players in such a game is just to use their optimal strategies for \( A \). In this case the payoff is \( V + c \), and so this is the value of \( B \).

**Answer 2:** Let \( p, q \) be a pair of optimal strategies for \( A \). Then, from the definition, we get that

\[
\min(p^T A) = V = \max(A q)
\]

(where min and max here represent the min and max entry of a vector).

Now we have

\[
\min(p^T B) = \min(p^T (A + C)) = \min(p^T A + p^T C)
\]

Note that since \( p \) is a probability vector, every entry of the vector \( p^T C \) is just \( c \). So

\[
\min(p^T B) = \min(p^T A) + c = V + c
\]

Similarly,

\[
\max(B q) = V + c
\]

From the definition of value, we get that \( V + c \) is the value of \( B \), with optimal strategies \( p \) and \( q \).

(b) Let \( A \) be the matrix given in the question. Notice that

\[
\begin{bmatrix}
2 & 1 & 0 & -1 \\
3 & 2 & 1 & 0 \\
4 & 3 & 2 & 1 \\
5 & 4 & 3 & 2
\end{bmatrix} = \begin{bmatrix}
0 & -1 & -2 & -3 \\
1 & 0 & -1 & -2 \\
2 & 1 & 0 & -1 \\
3 & 2 & 1 & 0
\end{bmatrix} + \begin{bmatrix}
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2
\end{bmatrix}
\]
The first matrix on the right hand side is skew-symmetric, so has value 0. From (a), we get that

\[ \text{Val}(A) = 0 + 2 = 2 \]