Section 1.4: The Matrix Equation $Ax = b$

- This section is about solving the “matrix equation” $Ax = b$, where $A$ is an $m \times n$ matrix and $b$ is a column vector with $m$ entries (both given in the question), and $x$ is an unknown column vector with $n$ entries (which we are trying to solve for). The first thing to know is what $Ax$ means: it means we are multiplying the matrix $A$ times the vector $x$. How do we multiply a matrix by a vector? We use the “row times column” rule, see the bottom of page 38 for examples.

- Solving $Ax = b$ is the same as solving the system described by the augmented matrix $[A|b]$.

- $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$.

- Theorem 4 is very important, it tells us that the following statements are either all true or all false, for any $m \times n$ matrix $A$:
  
  (a) For every $b$, the equation $Ax = b$ has a solution.
  
  (b) Every column vector $b$ (with $m$ entries) is a linear combination of the columns of $A$.

  (c) The columns of $A$ span $\mathbb{R}^m$ (this is just a restatement of (b), once you know what the word “span” means).

  (d) $A$ has a pivot in every row.

This theorem is useful because it means that if we want to know if $Ax = b$ has a solution for every $b$, we just need to check if $A$ has a pivot in every row. **Note:** If $A$ does not have a pivot in every row, that does not mean that $Ax = b$ does not have a solution for some given vector $b$. It just means that there are some vectors $b$ for which $Ax = b$ does not have a solution.

- Finally, it is very useful to know that multiplying a vector by a vector has the following nice properties:

  (a) $A(u + v) = A(u) + A(v)$, for vectors $u, v$

  (b) $A(cu) = cA(u)$, for vectors $u$ and scalars $c$.

Section 1.5: Solution Sets of Linear Systems

- A homogeneous system is one that can be written in the form $Ax = 0$. Equivalently, a homogeneous system is any system $Ax = b$ where $x = 0$ is a solution (notice that this means that $b = 0$, so both definitions match). The solution $x = 0$ is called the trivial solution. A solution $x$ is non-trivial is $x \neq 0$.

- The homogeneous system $Ax = 0$ has a non-trivial solution if and only if the equation has at least one free variable (or equivalently, if and only if $A$ has a column with no pivots).

- **Parametric vector form:** Let’s say you have found the solution set to a system, and the free variables are $x_3, x_4, x_5$. Then to write the solution set in ‘parametric vector form’ means to write the solution as

\[
x = p + x_3u + x_4v + x_5w
\]

where $p, u, v, w$ are vectors with numerical entries. A method for writing a solution set in this form is given on page 46.
Section 1.7: Linear Independence

- Like everything else in linear algebra, the definition of linear independence can be phrased in many different equivalent ways. \( \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p \) are linearly independent if any of the following equivalent statements are true:

  (a) the vector equation \( x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \cdots + x_2 \mathbf{v}_2 = 0 \) has only the trivial solution

  (b) none of the vectors \( \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p \) are a linear combination of the others

  (c) if we put the vectors together as columns of the matrix \( A \), then the system \( Ax = 0 \) has only the trivial solution

  (d) if we put the vectors together as columns of the matrix \( A \), then \( A \) has a pivot in every column

- If vectors aren’t linearly independent, then they are linearly dependent. This means that (at least) one of the vectors is a linear combination of the rest. Note: This does not mean that all of the vectors are linear combinations of the others. See the following exercise.

Exercise 1: Find three vectors in \( \mathbb{R}^3 \) that are linearly dependent, but where the third vector is not a linear combination of the first two.

Method to check linear (in)dependence: If we want to check if a set of given vectors is linearly independent, put them together as columns of a matrix, and then row reduce the matrix. If there is a pivot in every column, then they are independent. Otherwise, they are dependent.

Exercise 2 (1.7.1): Check if the following vectors are linearly independent:

\[
\begin{bmatrix}
5 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
7 \\
2 \\
-6
\end{bmatrix}, \begin{bmatrix}
9 \\
4 \\
-8
\end{bmatrix}
\]

Theorem 9: Any set containing the zero vector is linearly dependent. This follows immediately from the method above, because if one of the columns is zero, there can’t be a pivot in every column (there are other easy ways to prove this theorem also, see the book for example).

Theorem 8: If we have \( p \) vectors, each with \( n \) entries, and \( p > n \), then these vectors have to be linearly dependent. (This follows from the method above too, because if there are more columns than rows, there can’t be a pivot in every column).

Exercise 3: Find 2 vectors in \( \mathbb{R}^5 \) that are linearly dependent. Notice that this means that if \( p \leq n \) in the theorem above, then the vectors might be dependent or independent.