Numbered homework questions are from *A first course in abstract algebra* by J. B. Fraleigh.

**Homework 1 Due Jan 20**

p55 #6, 8
p56 #23, 26, 27
p57 #36(c)
p66 #18, 19
p133 #4, 6, 10, 18

**Homework 2 Due Jan 27**

p134 #20, 22, 26, 34, 40
p142 #2, 3, 4, 10, 12

**Homework 3 Due Feb 3**

p310 # 1, 2(a,b,c)

(3) Let \( R[x] \) be the additive group of polynomials over \( \mathbb{R} \) (polynomials with coefficients that are real numbers). Let \( \varphi : \mathbb{R}[x] \rightarrow \mathbb{R}[x] \) be given by \( \varphi(f) = f'' \). Check that \( \varphi \) is a homomorphism. Find \( \ker(\varphi) \).

(4) Let \( \mathbb{Z}[x] \) be the additive group of polynomials with integer coefficients. Let \( H \) be the subgroup of \( \mathbb{Z}[x] \) that consists of polynomials with 0 constant term. Prove that \( \mathbb{Z}[x]/H \) is a cyclic group.

**Homework 4 Due Feb 10**

p175 #10, 12, 13, 15, 16, 18, 19, 20, 24, 25, 26

**Homework 5 Due Feb 17**

p182 # 14

(2) Describe all ring homomorphisms

\[
f : \mathbb{Z}_{10} \rightarrow \mathbb{Z}_{15}
\]

such that \( f(1) = 1 \)

(3) Find all units in \( R = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \)
(4) Let \( R = 2\mathbb{Z}_{18} \). Is \( R \) an integral domain? Is \( R \) a field? Justify your answers.

(5) Let \( R = \{a + bi : a, b \in \mathbb{Z}_5, i^2 = -1\} \). Is \( R \) an integral domain? Is it a field?

(6) Find the multiplicative inverse of 11 in \( \mathbb{Z}_{13} \).

(7) Let \( R \) consist of matrices \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \), where \( a, b \) are real numbers. Describe the units in \( R \).

(8) Find all ideals in \( \mathbb{Z}_{12} \).

(9) Find all ideals in \( \mathbb{Z}_{15} \).

(10) Show that the ring of continuous functions from \([0, 1]\) to \( \mathbb{R} \) is not an integral domain.

**Homework 6 Due Feb 24**

(1) List all the elements in the subring of \( \mathbb{Z}_{12} \) generated by 4 and 6.

(2) Let \( R \) be the ring of all \( 2 \times 2 \) matrices over \( \mathbb{Z} \). Let \( S \) be the set of triangular matrices, i.e.

\[
S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}
\]

Is \( S \) a subring of \( R \)? Is \( S \) an ideal of \( R \)? Justify your answers.

(3) Let \( R = \mathbb{Z}[x] \) and let \( S \) be the subring of \( R \) generated by \( x^3 \) and \( x^5 \). Is \( S \) an ideal of \( R \)?

(4) Let \( R \) be the ring of all continuous functions from \([0, 1]\) to \( \mathbb{R} \).

For an arbitrary function \( f \in R \) define a function \( f' \in R \) by the following rule:

\[
f'(x) := \int_0^x f(t)dt, 0 \leq x \leq 1
\]

Is the mapping:

\[
R \rightarrow R
f \mapsto f'
\]

a homomorphism?
(5) Let $f : \mathbb{Z}_{30} \rightarrow \mathbb{Z}_{10}$ be a mapping such that $f(i) = i \pmod{10}$, $0 \leq i \leq 29$.

Is $f$ a homomorphism? If yes, find $\ker(f)$.

**Homework 7 Due Mar 3**

p243 #3, 4

p252-253 #2, 4, 5, 6, 15, 16, 17, 18

**Homework 8 Due Mar 10**

(1) Let $R = \mathbb{Z} + \mathbb{Z}i$ where $i^2 = -1$, and let $I = \{a + bi : 2 \text{ divides } a \text{ and } 2 \text{ divides } b\}$

(a) Show that $I$ is an ideal of $R$.

(b) Show that $I$ is not a maximal ideal of $R$

(2) Show that $\mathbb{Z} \oplus 5\mathbb{Z}$ is a maximal ideal of $\mathbb{Z} \oplus \mathbb{Z}$.

(3) Consider the polynomials:

$$f(x) = 2x^5 + 2x^4 + x^2 + 2 \quad g(x) = x^3 + 2x^2 + 2$$

in $\mathbb{Z}_3[x]$. Divide $f(x)$ by $g(x)$ with a remainder.

This means find polynomials $q(x)$ (the quotient) and $r(x)$ (the remainder), such that:

$$f(x) = q(x)g(x) + r(x)$$

where $\deg(r(x)) < \deg(g(x))$

(4) Determine whether each of the following polynomials is irreducible over each of the indicated fields:

(a) $x^2 + x + 2$ over $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$.

(b) $x^2 + x - 2$ over $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

(c) $x^3 - x^2 + 2x + 2$ over $\mathbb{Z}_3$, $\mathbb{Z}_5$ and $\mathbb{Z}_7$.

(5) Write the polynomial $x^4 + x^3 + x^2 + 2x + 3$ over $\mathbb{Z}_5$ as a product of irreducible polynomials.

(6) Determine whether the given set of vectors is a basis for $F^3$ over each of the indicated fields $F$:

(a) $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ over $\mathbb{Z}_2$ and $\mathbb{Z}_5$
(b) \((-1, 1, 3), (2, -3, 1), (5, -2, 0)\) over \(\mathbb{Z}_3\) and \(\mathbb{Z}_5\)

**Homework 9** Due Mar 17

1. Is \(\mathbb{Z}_5[\sqrt{3}]\) a field?
2. Is \(\mathbb{Z}_{11}[x]/\langle x^2 + x + 1 \rangle\) a field?
3. Is the ideal \(5\mathbb{Z}[i]\) maximal in \(\mathbb{Z}[i]\)?
4. Construct a field of order 125.
5. Let \(R = \mathbb{Z}_5[x], I = (x^2 + 2)R\). Find the multiplicative inverse of \((x+1)+I\) in \(R/I\).
6. Let \(V = \mathbb{Z}_3^2\) and \(W = \{(a, b, c) \in V : a^3 + b + c = 0\}\). Is \(W\) a subspace of \(V\)?