Midterm 2 solutions

Roman Kitsela

February 27, 2016

Question 1.

Proof. No, $\mathbb{Z}_5[i]$ is not even an integral domain:

$$(2 + i)(2 - i) = 0$$

\qed

Question 2.

Proof. No, as a counter example:

$1 + i \in A$, $i \in \mathbb{Z}[i]$ but $i(1 + i) = -1 + i \not\in A$ (Since $1 \not\equiv -1 \pmod{3}$).

\qed

Question 3.

Proof.

$$3x^4 - x + 1 = (2x^2 + 2x + 1)(-2x^2 + 2x - 1) + (6x + 2) \text{ in } \mathbb{Z}_7[x]$$

So quotient $-2x^2 + 2x - 1$, remainder $6x + 2$

\qed

Question 4.

Proof. Follow the example in the homework.

In $\mathbb{Z}[i]/\langle 2 - i \rangle$ we have $2 - i = 0 \implies 2 = i$.

So $(2)^2 = (i)^2 \implies 4 = -1$

That means that we have 5 potential coset representatives: $\{0, 1, 2, 3, 4\}$.

We check the order of $1\langle 2 - i \rangle$ in $\mathbb{Z}[i]/\langle 2 - i \rangle$:

The order is either 1 or 5. Suppose it has order 1:

i.e $1 \in \langle 2 - i \rangle \implies (2 - i)(a + bi) = 1$ has an integer solution.

But this reduces to $5b = 1$ so no solution is possible.

Hence 1 has order 5, and so all the representatives above are distinct. $\mathbb{Z}[i]/\langle 2 - i \rangle$ has 5 elements.

\qed
Question 5.

Proof. We want to find \(a, b\) in \(\mathbb{Z}_7\) such that:

\[(2x + 3)(ax + b) = 1\]

in \(\mathbb{Z}_7[x]/(x^2 + 1)\). We will use the fact that \(x^2 + 1 = 0\) (i.e. \(x^2 = -1\)) in the quotient ring

\[
\text{LHS} = 2ax^2 + (2b + 3a)x + 3b \\
= 2a(-1) + (2b + 3a)x + 3b \\
= (2b + 3a)x + (3b - 2a) \\
= 1
\]

Split this up into two equations by comparing components:

\[2b + 3a \equiv 0 \pmod{7}\]
\[3b - 2a \equiv 1 \pmod{7}\]

This solves to give:

\[a \equiv 2 \pmod{7}\]
\[b \equiv 4 \pmod{7}\]

A quick check confirms this is the correct inverse.

\(\square\)