Practice Questions for Final

1. Suppose \( A \) is an \( m \times n \) matrix and \( B \) is a \( n \times p \) matrix.

   If \( B \) is invertible and \( AB = 0 \), what can you conclude about \( A \)?

   Is this still true if \( B \) is not assumed to be invertible? Explain

2. Show that the transformation \( T \) defined by

   \[
   T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \\
   (x, y) \mapsto (x - 2y, x - 3, 2x - 5y)
   \]

   is not linear.

3. Consider the following matrix

   \[
   A = \begin{bmatrix}
   1 & 2 & 3 & -4 & 8 \\
   1 & 2 & 0 & 2 & 8 \\
   2 & 4 & -3 & 10 & 9 \\
   3 & 6 & 0 & 6 & 9
   \end{bmatrix}
   \]

   (a) Find a basis for \( \text{Col}(A) \)

   (b) Find a basis for \( \text{Null}(A) \)

   (c) Find a basis for \( \text{Row}(A) \)

   (d) What is Rank\((A)\)?

4. Suppose that \( \{v_1, \cdots, v_n\} \) is a linearly dependent set of vectors

   (a) Is it true that \( \{v_1, \cdots, v_n, v_{n+1}\} \) must also be linearly dependent
       (for any vector \( v_{n+1} \))

   (b) If \( \{v_1, \cdots, v_n\} \) also spans a vector space \( V \), what can you say
       about the dimension of \( V \)?

5. Let

   \[
   A = \begin{bmatrix}
   1 & 0 & -2 \\
   -3 & 1 & 4 \\
   2 & -3 & 4
   \end{bmatrix}
   \]

   Determine whether \( A \) is invertible and find \( A^{-1} \) if it is.
6. Let 
\[ \mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix} \]

Is \( \mathbf{u} \) in \( \text{Col}(\mathbf{A}) \)?
What is \( \text{Rank}(\mathbf{A}) \)?
Find a basis for \( \text{Null}(\mathbf{A}) \)

7. Describe the solution set of the following system of equations (write your answer in vector parametric form).
\[
\begin{align*}
    x_1 + 2x_2 - 3x_3 &= 5 \\
    2x_1 + x_2 - 3x_3 &= 13 \\
    -x_1 + x_2 &= -8
\end{align*}
\]

Compare this with the solution set to
\[
\begin{align*}
    x_1 + 2x_2 - 3x_3 &= 0 \\
    2x_1 + x_2 - 3x_3 &= 0 \\
    -x_1 + x_2 &= 0
\end{align*}
\]

8. Consider the following matrix
\[ \mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix} \]

Find a basis for \( \text{Null}(\mathbf{A}) \), \( \text{Col}(\mathbf{A}) \), \( \text{Row}(\mathbf{A}) \), \( \text{Col}(\mathbf{A}^T) \), \( \text{Row}(\mathbf{A}^T) \)

**Hint:** You might find the work from the previous question helpful

9. Let
\[ \mathbf{A} = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} \]

Find \( \mathbf{A}^{-1} \)
Write down $\text{Rank}(A)$ and $\text{dim}(\text{Null}(A))$ without doing any calculations

Use $A^{-1}$ to solve the following system of equations without using row reduction

$$
\begin{align*}
14x_1 + 4x_2 + 2x_3 &= 4 \\
0x_1 + 6x_2 - 2x_3 &= 8 \\
-6x_1 + 8x_2 - 4x_3 &= 2
\end{align*}
$$

10. Consider the following two matrices

$$
A = \begin{bmatrix}
1 & 4 & 8 & -3 & -7 \\
-1 & 2 & 7 & 3 & 4 \\
-2 & 2 & 9 & 5 & 5 \\
3 & 6 & 9 & -5 & -2
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
1 & 4 & 8 & 0 & 5 \\
0 & 2 & 5 & 0 & -1 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Matrix $B$ is a REF of $A$.

(a) Find a basis for $\text{Col}(A)$, what is $\text{rank}(A)$?
(b) Find a basis for $\text{Null}(A)$, what is the dimension of $\text{Null}(A)$?
(c) Find a basis for $\text{Col}(A^T)$

11. (a) If $F$ is a $5 \times 5$ matrix whose columns do not span $\mathbb{R}^5$, what can you say about $\text{dim}(\text{Null}(F))$?
(b) If $B$ is a $5 \times 5$ matrix and $\text{Null}(B)$ contains a non-zero vector, what can be said about $\text{Rank}(B)$?

12. Consider the following set of equations

$$
\begin{align*}
x + hy &= 2 \\
4x + 8y &= k
\end{align*}
$$

Find possible values of $h$ and $k$ such that the following system has

(a) No solution
(b) A unique solution
(c) Infinitely many solutions
13. (a) Find an equation involving $a$, $b$ and $c$ such that the following set of equations always has an solution.

\[
\begin{align*}
x - 4y + 7z &= a \\
3y - 5z &= b \\
-2x + 5y - 9z &= c
\end{align*}
\]

(b) Let

\[
A = \begin{bmatrix} 1 & -4 & 7 \\ 0 & 3 & -5 \\ -2 & 5 & -9 \end{bmatrix}
\]

Calculate $\text{Rank}(A)$ and $\dim(\text{Nul}(A))$

(c) Is $A$ invertible? If so find $A^{-1}$, if not explain why not

14. Let

\[
A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}
\]

Explain why the rank of $A$ is 3.

Do the columns of $A$ span $\mathbb{R}^3$? Why or why not?

15. Find and compare the solution sets of

\[
x + 5y - 3z = 0
\]

and

\[
x + 5y - 3z = -2
\]

Explain why the solution set of the first equation describes a 2-dimensional subspace of $\mathbb{R}^3$

Explain why the solution set of the second equation is not a subspace of $\mathbb{R}^3$

16. Consider the following vectors

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}
\]
(a) For which values of $h$ is $v_3$ in $Span\{v_1, v_2\}$?
(b) For which values of $h$ is the set $\{v_1, v_2, v_3\}$ linearly dependent.

17. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Suppose $\{u, v\}$ is linearly independent, but $\{T(u), T(v)\}$ is linearly dependent.

Show that $T(x) = 0$ must have a non-trivial solution.

(Hint: Use the fact that $c_1 T(u) + c_2 T(v) = 0$ has a solution for $c_1, c_2$ not both zero)

18. (a) $A$ is a $4 \times 4$ matrix whose columns do not span $\mathbb{R}^4$.
Is $A$ invertible? Why or why not?
(b) $B$ is a $7 \times 7$ matrix whose columns are linearly independent and $y$ is an arbitrary vector in $\mathbb{R}^7$.
Does the matrix equation $Bx = y$ always have a solution? Why or why not?

19. $T$ is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$$

(a) Find the standard matrix of $T$
(b) Find a formula for $T^{-1}$

20. Let

$$A = \begin{bmatrix}
3 & -1 & -3 & -1 & 8 \\
3 & 1 & 3 & 0 & 2 \\
0 & 3 & 9 & -1 & -4 \\
6 & 3 & 9 & -2 & 6
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
3 & -1 & -3 & 0 & 6 \\
0 & 2 & 6 & 0 & -4 \\
0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Given that $B$ is a REF of $A$

(a) Find a basis for $Col(A)$, what is rank($A$)?
(b) Find a basis for $Nul(A)$, what is the dimension of $Nul(A)$?
(c) Find a basis for $Col(A^T)$
21. Let

\[
H = \left\{ \begin{bmatrix} a + 2b - d + 3e \\ -a - 3b - c + 4d - 7e \\ -2a - b + 3c - 7d + 6e \\ 3a + 4b - 2c + 7d - 9e \end{bmatrix} : a, b, c, d, e \in \mathbb{R} \right\}
\]

Find a basis for \( H \).

22. Let \( V \) be the first quadrant in the \( xy \)-plane

\[
V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}
\]

(1) If \( u \) and \( v \) are vectors in \( V \), is \( u + v \) in \( V \)? Why?

(b) Find a vector \( u \) in \( V \) and a scalar \( c \) such that \( cu \) is \textbf{not} in \( V \).

Is \( V \) a subspace of \( \mathbb{R}^2 \)?

23. Let \( W \) be the union of the first and third quadrants in the \( xy \)-plane

\[
Q = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}
\]

(2) If \( u \) is in \( W \), and \( c \) is any scalar, is \( cu \) in \( W \)? Why?

(b) Find vectors \( u \) and \( v \) in \( W \) such that \( u + v \) is \textbf{not} in \( W \).

Is \( W \) a subspace of \( \mathbb{R}^2 \)?

24. In each of the following parts the set \( W \) be the set of all vectors of the form shown, where \( a \), \( b \) and \( c \) represent arbitrary real numbers. In each case, either find a set of vectors which spans \( W \) or give an example to show \( W \) is not a vector space

(a) Let

\[
W_1 = \left\{ \begin{bmatrix} 1 \\ 3a - 5b \\ 3b + 2a \end{bmatrix} : a, b \in \mathbb{R} \right\}
\]

Is \( W_1 \) a subspace of \( \mathbb{R}^3 \)? If so, find a set of basis vectors. If not, explain why not.
(b) Let

\[ W_2 = \left\{ \begin{bmatrix} 4a \\ 0 \\ a + 3b + c \\ 3b - 2c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \]

Is \( W_2 \) a subspace of \( \mathbb{R}^4 \)? If so, find a set of basis vectors. If not, explain why not.

25. Suppose that \( T \) is a one-to-one transformation.

Show that if \( \{T(v_1), \ldots, T(v_p)\} \) is linearly dependent, then \( \{v_1, \ldots, v_p\} \) must be linearly dependent.

26. Consider the polynomials \( p_1(t) = 1 + t^2 \) and \( p_2(t) = 1 - t^2 \). Is \( \{p_1, p_2\} \) a linearly independent set in \( \mathbb{P}_3 \)? Why or why not?

Write down a basis for \( \mathbb{P}_3 \).

27. (a) Show that the set \( B = \{1 - t^2, t - t^2, 2 - t - t^2\} \) is not a basis for \( \mathbb{P}_2 \).

(b) Explain why \( C = \{1 - t^2, t - t^2, 2 - t - 2t^2\} \) is a basis. (Be clear about what is different between your answers to (a) and (b))

(c) Find the co-ordinate vector of \( p(t) = 3 - 4t^2 \) relative to \( C \).

(d) Find the polynomial with co-ordinate vector

\[ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \]

relative to \( C \).

28. Show that \( \{1, 2t, -2 + 4t^2, -12t + 8t^3\} \) is a basis for \( \mathbb{P}_3 \).

29. Let

\[ A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix} \]

(a) Find the eigenvalues of \( A \).

(b) Find a basis for each eigenspace of \( A \).

(c) (If possible) diagonalize \( A \).
30. Let
\[
A = \begin{bmatrix}
2 & 4 & 3 \\
-4 & -6 & -3 \\
3 & 3 & 1
\end{bmatrix}
\]
Diagonalize \( A \) if possible. If not, explain why not.

31. Let
\[
A = \begin{bmatrix}
2 & 0 & 0 \\
2 & 2 & 0 \\
2 & 2 & 2
\end{bmatrix}
\]
(a) What is the characteristic polynomial of \( A \)?
(b) Find the eigenvalues of \( A \)
(c) Find a basis for each eigenspace of \( A \)
(d) Diagonalize \( A \) (if possible)

32. (a) \( A \) is a 5 \( \times \) 5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is \( A \) diagonalizable? Why?
(b) \( A \) is a 7 \( \times \) 7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that \( A \) is not diagonalizable?

33. Let
\[
B = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 2 \\
1 & 2 & 1
\end{bmatrix}
\]
Compute \( \det(B^5) \)

34. \( A \) and \( B \) be both 3 \( \times \) 3 matrices with \( \det(A) = 4 \) and \( \det(B) = -3 \).
Compute the following determinants if possible or state "cannot determine" if not.
(a) \( \det(AB) \)
(b) \( \det(A + B) \)
(c) \( \det(5A) \)
(d) \( \det(A^{-1}) \)
(e) \( \det(B^{-1} + 4A) \)
(f) $\det(A^3)$
(g) $\det(B^T)$

35. Let

$$A = \begin{bmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{bmatrix}$$

Calculate $\det(A)$ in two different ways

36. Let

$$B = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}$$

Calculate $\det(B)$ in two different ways

37. Show that if $A$ is invertible, then:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

38. Suppose that if $A$ is diagonalizable, then $\det(A)$ is the product of the eigenvalues of $A$

   **Hint:** Start by writing down what it means for $A$ to be diagonalizable, then take the determinant of both sides of the equation and use the properties of determinants.

39. Suppose $B$ is a matrix such that $B^2 = 0$ (i.e. $B^2$ is the 0 matrix).

   Show that the only eigenvalue of $B$ is 0

40. Let

   $$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$$

   (a) Find $x$ given $[x]_\mathcal{B} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

   (b) Find $[x]_\mathcal{B}$ given $x = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$
41. Let 
\[ B = \left\{ \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\} \text{ and } C = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \]

(a) Find \( P_{C \leftarrow B} \)
(b) Find \( P_{B \leftarrow C} \)

(c) Given that \( x \) has coordinates \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) relative to \( B \) (i.e. \( [x]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)), write down the coordinates of \( x \) relative to \( C \).

42. Let 
\[ B = \left\{ \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\} \text{ and } C = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix} \right\} \]

(a) Find \( P_{C \leftarrow B} \)
(b) Find \( P_{B \leftarrow C} \)

(c) How are your answers to (a) and (b) related?

43. Suppose a vector \( y \) is orthogonal to vectors \( u \) and \( v \). Show that \( y \) is orthogonal to any vector in \( \text{span}\{u, v\} \).

**Hint:** An arbitrary vector \( z \) in \( \text{span}\{u, v\} \) has the form \( z = c_1u + c_2v \).
Show that \( y \) is orthogonal to such a \( z \).

44. Consider the following set of vectors 
\[ S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \right\} \]

(a) Show that \( S \) is an orthogonal set.
(b) Write down an **orthonormal** set corresponding to \( S \)

45. Consider the following set of vectors 
\[ S = \left\{ \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \]

(a) Show that \( S \) is an orthogonal basis for \( \mathbb{R}^3 \).
(b) Express
\[
x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}
\]
as a linear combination of vectors in \( S \)

46. Let
\[
y = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}
\]

(a) Compute the orthogonal projection of \( y \) onto the line through \( u \) and the origin.

(b) Write \( y \) as a sum of a vector in \( \text{span}\{u\} \) and a vector orthogonal to \( u \).

(c) Find the distance from \( y \) to the line through \( u \) and the origin.