Practice Questions for Final

1. Suppose A is an $m \times n$ matrix and B is a $n \times p$ matrix.

   If B is invertible and $AB = 0$, what can you conclude about A?

   Is this still true if B is not assumed to be invertible? Explain

2. Show that the transformation T defined by

$$\begin{aligned} T : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ (x, y) &\longmapsto (x - 2y, x - 3, 2x - 5y) \end{aligned}$$

   is not linear.

3. Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}$$

   (a) Find a basis for Col(A)

   (b) Find a basis for Null(A)

   (c) Find a basis for Row(A)

   (d) What is Rank(A)?

4. Suppose that $\{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ is a linearly dependent set of vectors

   (a) Is it true that $\{\mathbf{v_1}, \cdots, \mathbf{v_n}, \mathbf{v_{n+1}}\}$ must also be linearly dependent (for any vector $\mathbf{v}_{n+1}$)

   (b) If $\{\mathbf{v_1}, \cdots, \mathbf{v_n}\}$ also spans a vector space V, what can you say about the dimension of V?

5. Let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

   Determine whether A is invertible and find A^{-1} if it is.

6. Let

$$\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 5 & -1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

Is $\mathbf{u}$ in $\text{Col}(A)$?

What is $\text{Rank}(A)$?

Find a basis for $\text{Null}(A)$

7. Describe the solution set of the following system of equations (write your answer in vector parametric form).

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 5 \\ 2x_1 + x_2 - 3x_3 &= 13 \\ -x_1 + x_2 &= -8 \end{aligned}$$

Compare this with the solution set to

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

8. Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix}$$

Find a basis for $\text{Null}(A)$, $\text{Col}(A)$, $\text{Row}(A)$, $\text{Col}(A^T)$, $\text{Row}(A^T)$

Hint: You might find the work from the previous question helpful

9. Let

$$A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

Find A^{-1}

Write down Rank(A) and dim(Null(A)) **without doing any calculations**

Use A^{-1} to solve the following system of equations **without using row reduction**

$$\begin{aligned} 14x_1 + 4x_2 + 2x_3 &= 4 \\ 0x_1 + 6x_2 - 2x_3 &= 8 \\ -6x_1 + 8x_2 - 4x_3 &= 2 \end{aligned}$$

10. Consider the following two matrices

$$A = \begin{bmatrix} 1 & 4 & 8 & -3 & -7 \\ -1 & 2 & 7 & 3 & 4 \\ -2 & 2 & 9 & 5 & 5 \\ 3 & 6 & 9 & -5 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 8 & 0 & 5 \\ 0 & 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix B is a REF of A.

(a) Find a basis for Col(A), what is rank(A)?

(b) Find a basis for Null(A), what is the dimension of Null(A)?

(c) Find a basis for Col(A^T)

11. (a) If F is a 5×5 matrix whose columns do not span $\mathbb{R}^5$, what can you say about dim(Null(F))?

(b) If B is a 5×5 matrix and Null(B) contains a non-zero vector, what can be said about Rank(B)?

12. Consider the following set of equations

$$\begin{aligned} x + hy &= 2 \\ 4x + 8y &= k \end{aligned}$$

Find possible values of h and k such that the following system has

(a) No solution

(b) A unique solution

(c) Infinitely many solutions

13. (a) Find an equation involving a, b and c such that the following set of equations always has an solution.

$$\begin{aligned} x - 4y + 7z &= a \\ 3y - 5z &= b \\ -2x + 5y - 9z &= c \end{aligned}$$

(b) Let

$$A = \begin{bmatrix} 1 & -4 & 7 \\ 0 & 3 & -5 \\ -2 & 5 & -9 \end{bmatrix}$$

Calculate Rank(A) and dim(Nul(A))

(c) Is A invertible? If so find A^{-1}, if not explain why not

14. Let

$$A = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}$$

Explain why the rank of A is 3.

Do the columns of A span $\mathbb{R}^3$? Why or why not?

15. Find and compare the solution sets of

$$x + 5y - 3z = 0$$

and

$$x + 5y - 3z = -2$$

Explain why the solution set of the first equation describes a 2-dimensional **subspace** of $\mathbb{R}^3$

Explain why the solution set of the second equation is not a subspace of $\mathbb{R}^3$

16. Consider the following vectors

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$$

(a) For which values of h is $\mathbf{v_3}$ in $Span\{\mathbf{v_1}, \mathbf{v_2}\}$?

(b) For which values of h is the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ linearly dependent.

17. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Suppose $\{\mathbf{u}, \mathbf{v}\}$ is linearly **independent**, but $\{T(\mathbf{u}), T(\mathbf{v})\}$ is linearly **dependent**.

    Show that $T(\mathbf{x}) = \mathbf{0}$ must have a non-trivial solution.

    (**Hint:** Use the fact that $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$ has a solution for c_1, c_2 not both zero)

18. (a) A is a 4×4 matrix whose columns **do not** span $\mathbb{R}^4$.
    Is A invertible? Why or why not?

    (b) B is a 7×7 matrix whose columns are linearly independent and $\mathbf{y}$ is an arbitrary vector in $\mathbb{R}^7$.
    Does the matrix equation $B\mathbf{x} = \mathbf{y}$ always have a solution? Why or why not?

19. T is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2)$$

    (a) Find the standard matrix of T

    (b) Find a formula for T^{-1}

20. Let

$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

    Given that B is a REF of A

    (a) Find a basis for Col(A), what is rank(A)?

    (b) Find a basis for Nul(A), what is the dimension of Nul(A)?

    (c) Find a basis for Col(A^T)

21. Let

$$H = \left\{ \begin{bmatrix} a + 2b - d + 3e \\ -a - 3b - c + 4d - 7e \\ -2a - b + 3c - 7d + 6e \\ 3a + 4b - 2c + 7d - 9e \end{bmatrix} : a, b, c, d, e \in \mathbb{R} \right\}$$

Find a basis for H

22. Let V be the first quadrant in the xy-plane

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\} \tag{1}$$

   (a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?

   (b) Find a vector $\mathbf{u}$ in V and a scalar c such that $c\mathbf{u}$ is **not** in V.

   Is V is a subspace of $\mathbb{R}^2$

23. Let W be the union of the first and third quadrants in the xy-plane

$$Q = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\} \tag{2}$$

   (a) If $\mathbf{u}$ is in W, and c is any scalar, is $c\mathbf{u}$ in W? Why?

   (b) Find vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $\mathbf{u} + \mathbf{v}$ is **not** in W.

   Is W is a subspace of $\mathbb{R}^2$

24. In each of the following parts the set W be the set of all vectors of the form shown, where a, b and c represent arbitrary real numbers. In each case, either find a set of vectors which spans W or give an example to show W is not a vector space

   (a) Let

$$W_1 = \left\{ \begin{bmatrix} 1 \\ 3a - 5b \\ 3b + 2a \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

   Is W_1 a subspace of $\mathbb{R}^3$? If so, find a set of basis vectors. If not, explain why not.

(b) Let

$$W_2 = \left\{ \begin{bmatrix} 4a \\ 0 \\ a+3b+c \\ 3b-2c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

Is W_2 a subspace of $\mathbb{R}^4$? If so, find a set of basis vectors. If not, explain why not.

25. Suppose that T is a one-to-one transformation.

    Show that if $\{T(\mathbf{v_1}), \cdots, T(\mathbf{v_p})\}$ is linearly dependent, then $\{\mathbf{v_1}, \cdots, \mathbf{v_p}\}$ must be linearly dependent.

26. Consider the polynomials $\mathbf{p_1}(t) = 1+t^2$ and $\mathbf{p_2}(t) = 1-t^2$. Is $\{\mathbf{p_1}, \mathbf{p_2}\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

    Write down a basis for $\mathbb{P}_3$

27. (a) Show that the set $\mathcal{B} = \{1-t^2, t-t^2, 2-t-t^2\}$ is **not** a basis for $\mathbb{P}_2$.

    (b) Explain why $\mathcal{C} = \{1-t^2, t-t^2, 2-t-2t^2\}$ **is** a basis. (Be clear about what is different between your answers to (a) and (b))

    (c) Find the co-ordinate vector of $\mathbf{p}(t) = 3-4t^2$ relative to $\mathcal{C}$

    (d) Find the polynomial with co-ordinate vector

    $$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

    relative to $\mathcal{C}$.

28. Show that $\{1, 2t, -2+4t^2, -12t+8t^3\}$ is a basis for $\mathbb{P}_3$

29. Let

    $$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

    (a) Find the eigenvalues of A

    (b) Find a basis for each eigenspace of A

    (c) (If possible) diagonalize A

30. Let

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Diagonalize A if possible. If not, explain why not.

31. Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$

(a) What is the characteristic polynomial of A?

(b) Find the eigenvalues of A

(c) Find a basis for each eigenspace of A

(d) Diagonalize A (if possible)

32. (a) A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

(b) A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is **not** diagonalizable?

33. Let

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Compute $det(B^5)$

34. A and B be both 3×3 matrices with $\det(A) = 4$ and $\det(B) = -3$.

Compute the following determinants if possible or state "cannot determine" if not.

(a) $\det(AB)$

(b) $\det(A + B)$

(c) $\det(5A)$

(d) $\det(A^{-1})$

(e) $\det(B^{-1} + 4A)$

(f) $\det(A^3)$

(g) $\det(B^T)$

35. Let

$$A = \begin{bmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{bmatrix}$$

Calculate $\det(A)$ in two different ways

36. Let

$$B = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}$$

Calculate $\det(B)$ in two different ways

37. Show that if A is invertible, then:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

38. Suppose that if A is diagonalizable, then $\det(A)$ is the product of the eigenvalues of A

Hint: Start by writing down what it means for A to be diagonalizable, then take the determinant of both sides of the equation and use the properties of determinants.

39. Suppose B is a matrix such that $B^2 = 0$ (i.e. B^2 is the 0 matrix).

Show that the only eigenvalue of B is 0

40. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$$

(a) Find $\mathbf{x}$ given

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(b) Find $[\mathbf{x}]_{\mathcal{B}}$ given

$$\mathbf{x} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

41. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

(a) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

(b) Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

(c) Given that **x** has coordinates $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ relative to $\mathcal{B}$ (i.e. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$), write down the coordinates of **x** relative to $\mathcal{C}$.

42. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} -6 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \end{bmatrix} \right\}$$

(a) Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

(b) Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$

(c) How are your answers to (a) and (b) related?

43. Suppose a vector **y** is orthogonal to vectors **u** and **v**. Show that **y** is orthogonal to any vector in span$\{\mathbf{u}, \mathbf{v}\}$

Hint: An arbitrary vector **z** in span$\{\mathbf{u}, \mathbf{v}\}$ has the form $\mathbf{z} = c_1\mathbf{u} + c_2\mathbf{v}$. Show that **y** is orthogonal to such a **z**.

44. Consider the following set of vectors

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(a) Show that $\mathcal{S}$ is an orthogonal set.

(b) Write down an **orthonormal** set corresponding to $\mathcal{S}$

45. Consider the following set of vectors

$$\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

(a) Show that $\mathcal{S}$ is an orthogonal basis for $\mathbb{R}^3$.

(b) Express

$$\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

as a linear combination of vectors in $\mathcal{S}$

46. Let

$$\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

(a) Compute the orthogonal projection of $\mathbf{y}$ onto the line through $\mathbf{u}$ and the origin.

(b) Write $\mathbf{y}$ as a sum of a vector in span$\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

(c) Find the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.