Math 18 – Lecture 20 + 21 (re-recording)

Determinant of a matrix is a scalar (ie a number) quantity that gives you some information about the matrix:

* whether the matrix is invertible
* Geometric information about the linear transformation corresponding to the given matrix.

The determinant is only defined for square matrices.

Determinant of a 2×2 matrix

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} := ab - bc$$

defined to be

eg) $\det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 1-(-1) = 2$

eg) $\det\begin{pmatrix} 2-\lambda & 1 \\ -3 & -1-\lambda \end{pmatrix}$

$$= (2-\lambda)(-1-\lambda) - (-3)$$

$$= \lambda^2 - 2\lambda + \lambda - 2 + 3$$

$$= \lambda^2 - \lambda + 1$$

Determinants of larger matrices

Given a matrix A $(n \times n)$ the submatrix denoted by

A_{ij} is obtained by ignoring the i^{th} row and j^{th} column from A

(ie A_{ij} is $(n-1) \times (n-1)$)

a_{11}

eg) $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

$A_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

$A_{12} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

$\boxed{\text{Def}^n}$ A_{ij} are called the $(i,j)^{th}$ minor matrices of A

Determinant Recursive Definition

For $n \geq 2$ the determinant of $A = [a_{ij}]$ is defined to be:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

(This is called expanding along the 1^{st} row)

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

$$= (-1)^{1+1} a_{11} \det(A_{11}) +$$

$$(-1)^{1+2} a_{12} \det(A_{12}) + \cdots$$

$$\cdots + (-1)^{n+1} a_{1n} \det(A_{1n})$$

minor matrices of A

entries along 1st row of A

pattern of + and - signs.

Example

entries along 1st ro

$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{pmatrix} \end{array}$$

$$\det(A) = \underbrace{+}_{(-1)^{1+1}} \underbrace{1}_{a_{11}} \cdot \det \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}}_{A_{11}}$$

$$+ \underbrace{(-1)}_{(-1)^{1+2}} \underbrace{-1}_{a_{12}} \cdot \det \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}}_{A_{12}}$$

$$+ \underbrace{(+1)}_{(-1)^{1+3}} \underbrace{0}_{a_{13}} \cdot \det \underbrace{\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}}_{A_{13}}$$

$$= (0-1) + (6-1) + 0$$

$$= 4$$

Remarks

It is actually to calculate the determinant by expanding along **any** row (or even column)

Example (different calculation)

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

I'll expand 2nd

$$\begin{pmatrix} 1 & 1 & 3 \end{pmatrix}$$

along column.

Applying formula above by expanding along 2^{nd} column we would get :

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

pattern of + and − in formula

$\det(A) =$

A_{12}

$$-(-1)\det\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

A_{22}

$$+(0)\det\begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$$

A_{32}

$$-(1)\det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$= +(6-1) + 0 - (1-0)$

$= 4$

Theorems concerning determinants

$\boxed{\text{Th}^{\underline{m}}\ 2}$ If A is triangular

Then: $\det(A) =$ product of its diagonal entries.

Upper Triangular

Has 0s below the main

diagonal i.e $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

Lower Triangular

Has 0s above the main diagonal i.e $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 3 \end{pmatrix}$

Remark

Matrices in REF are always triangular (upper triangular)

$\boxed{\text{Th}^{\underline{m}}\ 3}$ (Row Operations and Determinants)

Let A be a square matrix:

① If a multiple of a row of A is added to another row of A to form matrix B

Then: $\det(B) = \det(A)$

(ie no change to det.)

② If two rows of A are

interchanged to form B

Then : $\det(B) = -\det(A)$

③ If a row of A is multiplied by k to form matrix B, then:

$$\det(B) = k\det(A)$$

(This last property is easiest to think about as "factoring out" constants from rows)

The above theorems allows us to implement a strategy useful in calculating large matrix determinants...

1. Start with A and row reduce to REF

   * Keeping track of row operations

2. Calculate the determinant of the REF of A using Thm 2

③ Make adjustments to the determinant calculated in part ② to work out the determinant of A.

Example

$$A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$$

$\det(A) = |A|$

$$\det(A) = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

"Factor out" 2 from 1st row using part (3) of Thm 3

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

want to be 0

$R_2 - 3R_1$
$R_3 + 3R_1$
$R_4 - R_1$

(These don't affect determinant)

$$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ \dots \end{vmatrix}$$

$$\begin{vmatrix} 0 & -12 & 15 & 16 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$$\Big\downarrow R_3 + 4R_2$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

factored out another 2 from 3rd row

$$= 2(2)\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & -3 & 2 \end{vmatrix}$$

$\downarrow R_4 - R_3$

$$= 2(2)\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

all 0

matrix in REF

$$= 4(1 \cdot 3 \cdot -3 \cdot 1)$$

$$= -36$$

The above result can be summarized as follows:

# of row swaps

$$\det(A) = (-1)^{r} \cdot \det(U)$$

REF of A

$$\det(A) = \begin{cases} (-1)^r \cdot \begin{pmatrix} \text{Product of} \\ \text{pivots of } A \\ \text{in REF} \end{pmatrix} & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible} \end{cases}$$

(Why does the above depend on the invertibility of A?)

REF of A has pivots in every row and column if and only if A is invertible

(Invertible matrix thm)

The REF has a pivot in every row and column if its diagonal entries are <u>all</u> non-zero

In this case we apply Thm 2

In this case we apply Thm 2 to see that the determinant is a product of the pivots (along the diagonal necessarily)

otherwise, if the REF has a 0 on the diagonal

$\Rightarrow$ It cannot have a pivot in every row and colm

$\Rightarrow$ A is not invertible

We can summarize this as the following theorem:

$\boxed{\text{Th}^{\underline{m}}\ 4}$ A sq. matrix is invertible if and only if $\det(A) \neq 0$

$\boxed{\text{Th}^{\underline{m}}\ 5}$ $\det(A) = \det(A^T)$

$\boxed{\text{Th}^{\underline{m}}\ 6}$ $\det(AB) = \det(A)\det(B)$

§3.3 - Cramer's Rule, Volume Linear Transformations

Method used to solve $A\underline{x} = \underline{b}$

Notation

Given a matrix A $(n \times n)$ and $\underline{b}$ in $\mathbb{R}^n$

$A_i(\underline{b})$ means Matrix A with its ith column replaced by $\underline{b}$

Cramer's Rule

Given a matrix eq^n: $A\underline{x} = \underline{b}$

The components x_i of the solution $\underline{x}$ are given by:

$$\boxed{x_i = \frac{\det(A_i(\underline{b}))}{\det(A)}}$$

for $i = 1, 2, \dots, n$

Example

Let's solve:

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

$\downarrow$ rewrite as matrix eqⁿ

$$\underbrace{\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$A_1(\underline{b}) = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}$$

$$A_2(\underline{b}) = \begin{pmatrix} 3 & 6 \\ -5 & 8 \end{pmatrix}$$

$$\det(A) = 12 - 10 = 2$$

$$\det\left(A_1(\underline{b})\right) = 24 + 16 = 40$$

$$\det\left(A_2(\underline{b})\right) = 24 + 30 = 54$$

$$x_1 = \frac{40}{2} = 20$$

$$x_2 = \frac{54}{2} = 27$$

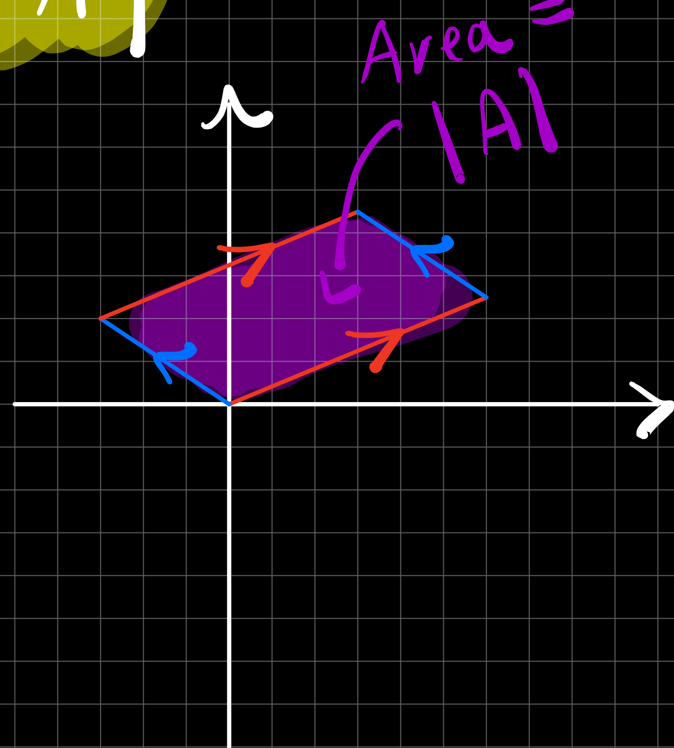
$$x_2 = \frac{\quad}{2} = 2$$

$$\left(\text{ie } \underline{x} = \begin{pmatrix} 20 \\ 27 \end{pmatrix}\right)$$

Determinants as areas and volumes

Thm 9 If A is 2×2, the area of the parallelogram determined by the columns of A is given by $|\det A|$

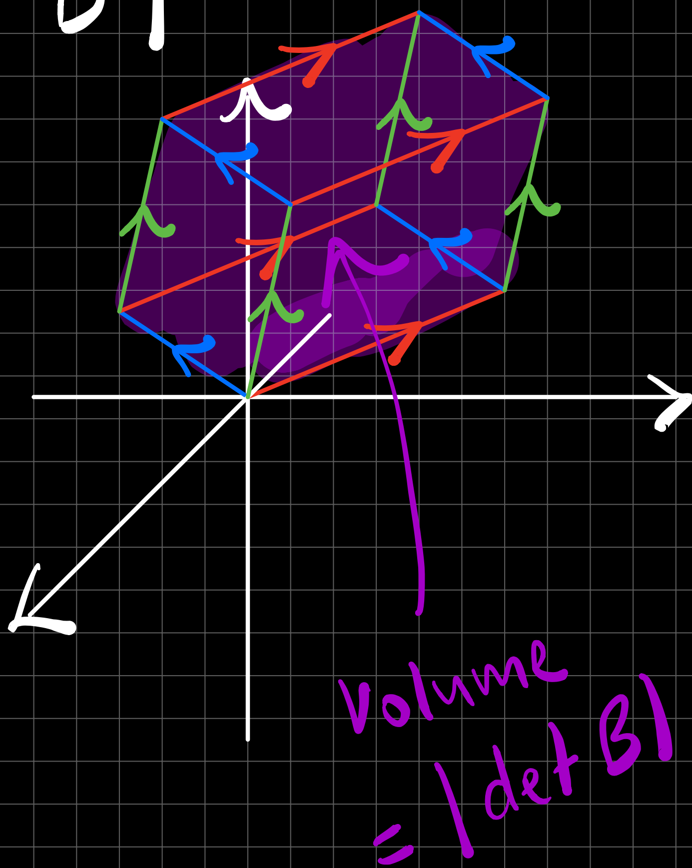
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



This generalizes to 3×3 matrices:

If B is 3×3, the volume of the parallelopiped determined by the columns of B is given by $|\det B|$

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



Determinants and linear transformations

Suppose A is 2×2 and S is a shape in $\mathbb{R}^2$.

If $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation whose std. matrix is A (ie $T(\underline{x}) = A\underline{x}$)

Then the area of $\boxed{T(S)}$

$= |\det(A)| \cdot$ Area of S

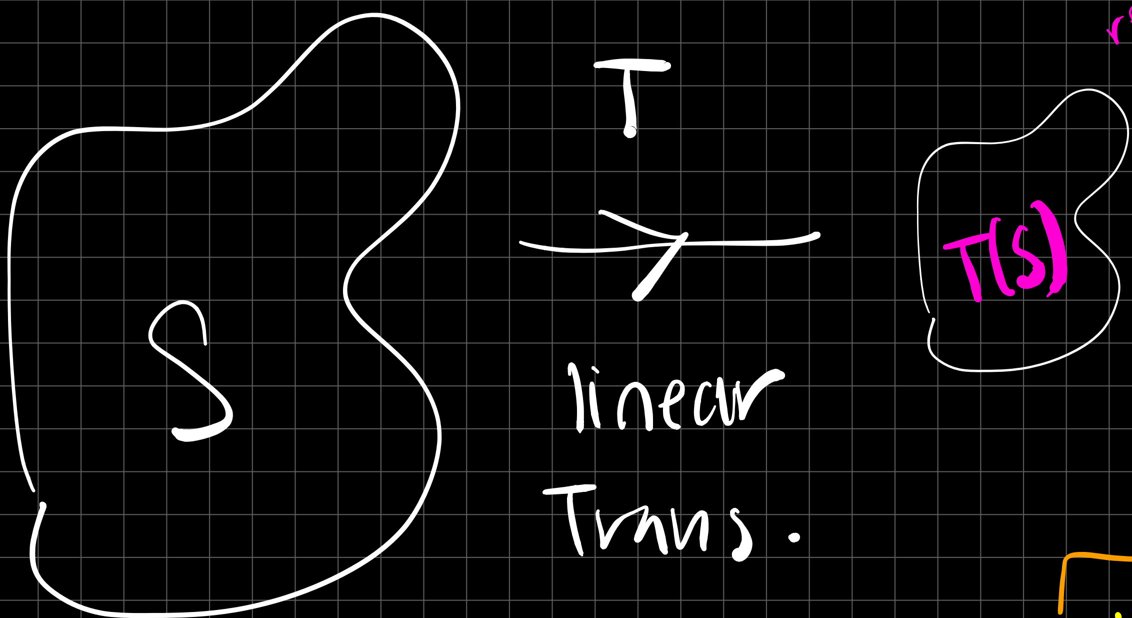
mapped

image of S

i.e. $|\det(A)|$ give a ratio for how the size of a shape changes when mapped by a linear transformation...

Total area of $T(S)$ = Area of S × det of A represents T

eg)



$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{1}{2}x\\ \frac{1}{2}y\end{pmatrix} = \overbrace{\begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}}^{A}\begin{pmatrix}x\\y\end{pmatrix}$$

T makes things smaller by a factor of $1/2$