Name: Answer Key at end Sec. No. ________________

1. Math 20F (Driver) Test #1: Friday, 10/15/2010

Directions: Please do not use the text, lecture notes, or calculators on this test. Write your solutions clearly and explain what you are doing – do not simply write answers down with no explanation unless explicitly instructed to do so. All problems are worth 10 points each.

1. Show that

$$C = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & -1 \\ 2 & 0 & -2 & -2 \end{bmatrix} \text{ is row equivalent to } \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

To get credit for this problem you **must** show your steps and explain what row operations you are doing at each stage!!

2. Describe the general solution to the system of equations

$$x_1 + 2x_2 + 3x_3 = 1,$$
$$3x_1 + 2x_2 + x_3 = -1,$$
$$2x_1 + 0x_2 - 2x_3 = -2.$$

3. Suppose that

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ (from Eq. (1).}$$

a: Describe the general solution to the equation $A\mathbf{x} = \mathbf{0}$.

b: Is it possible to solve the system of equations $A\mathbf{x} = \mathbf{b}$ independent of the choice of $\mathbf{b} \in \mathbb{R}^3$? Briefly explain your answer.

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4. Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and } \mathbf{a}_3 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}.$$

a: Are $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ linearly dependent? If yes express one of the three vectors as a linear combination of the other two.

b: Do $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ span $\mathbb{R}^3$?

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5. Answer the following question true or false. (No explanations need be given.)

a: For every 3×2 matrix A, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution, namely $\mathbf{x} = \mathbf{0}$. ________

b: The columns of a 3×2 matrix can never span all of $\mathbb{R}^3$. ________

c: If A is a 2×3 matrix then it is possible that the equation $A\mathbf{x} = \mathbf{0}$ has **only** the trivial solution, $\mathbf{x} = \mathbf{0}$. ________

6. Find all values of $\lambda \in \mathbb{R}$ so that the equation $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution, where

$$A := \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}.$$

7. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find: a) $T\left(\begin{bmatrix} 3\pi \\ 2\pi \end{bmatrix}\right)$ and b) $T\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$.

8. Let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ be three vectors in $\mathbb{R}^4$ and $A = [\mathbf{a}_1|\mathbf{a}_2|\mathbf{a}_3]$ – a 4×3 matrix. Answer the following questions true or false (no explanation need be given);

a: If there are real numbers x_1, x_2, and x_3 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = 0$, then $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a linearly dependent set. ________

b: It is possible that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ span $\mathbb{R}^4$. ________

c: It is possible that the equation $A\mathbf{x} = \mathbf{0}$ has no solutions. ________

9. Suppose that A is a 3×4 matrix such that $A \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $C = \text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$,

i.e. C is the reduced echelon form of A. Describe **all** solutions $\mathbf{x}$ to the equation $A\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

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10. Suppose A is a 3×4 matrix which is again row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Answer the following questions and give a **brief** reason for your answer.

a.: Do the columns of A span $\mathbb{R}^3$?

b: Are the columns of A linearly independent?

Test 1 Solutions

1.

$$C=\begin{bmatrix}1&2&3&1\\3&2&1&-1\\2&0&-2&-2\end{bmatrix}\to\begin{bmatrix}1&2&3&1\\0&-4&-8&-4\\0&-4&-8&-4\end{bmatrix}\begin{matrix}R_2\to R_2-3R_1\\R_3\to R_3-2R_1\end{matrix}$$

$$\to\begin{bmatrix}1&2&3&1\\0&-4&-8&-4\\0&0&0&0\end{bmatrix}\ R_3\to R_3-R_2$$

$$\to\begin{bmatrix}1&2&3&1\\0&1&2&1\\0&0&0&0\end{bmatrix}\ R_2\to-\tfrac{1}{4}R_2$$

$$\to\begin{bmatrix}1&0&-1&-1\\0&1&2&1\\0&0&0&0\end{bmatrix}\ R_1\to R_1-2R_2$$

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2. The matrix C is the augmented matrix for this system and therefore we may use the row reduced form of C to get an equivalent system which becomes;

$$x_1-x_3=-1 \text{ and } x_2+2x_3=1.$$

Now x_3 is a free variable and therefore

$$\begin{bmatrix}-1+x_3\\1-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}-1\\1\\0\end{bmatrix}+\begin{bmatrix}x_3\\-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}-1\\1\\0\end{bmatrix}+x_3\begin{bmatrix}1\\-2\\1\end{bmatrix}\text{ with } x_3 \text{ free}$$

describes all of the possible solutions. As a check observe that

$$\begin{bmatrix}1&2&3\\3&2&1\\2&0&-2\end{bmatrix}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\-1\\-2\end{bmatrix}\text{ and}$$

$$\begin{bmatrix}1&2&3\\3&2&1\\2&0&-2\end{bmatrix}\begin{bmatrix}1\\-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

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3. Since

$$[A|0] \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we know that x_3 is a free variable and $x_1 = x_3$ and $x_2 = -2x_3$ so the solution is

$$\begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ where } x_3 \text{ is free.}$$

b. Since rref(A) has a row of zeros, i.e. there is not a pivot in the last row it is **not** always possible to solve $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^3$. ■

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4. a. Since $A = [a_1|a_2|a_3] \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ has a free variable, the $\{a_1, a_2, a_3\}$ are linearly dependent and from Problems 3 we know that $1a_1 - 2a_2 + a_3 = 0$ so that

$$a_1 = 2a - a_3.$$

b. No they do not span $\mathbb{R}^3$ since again there is not a pivot in every row. See part 3b which is equivalent to this question. ■

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5. Answer the following question true or false. (No explanations need be given.)

a: For every 3×2 matrix A, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution, namely $\mathbf{x} = \mathbf{0}$. False

b: The columns of a 3×2 matrix can never span all of $\mathbb{R}^3$. True

c: If A is a 2×3 matrix then it is possible that the equation $A\mathbf{x} = \mathbf{0}$ has **only** the trivial solution, $\mathbf{x} = \mathbf{0}$. False

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6.

$$\begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 1-\lambda \\ 1-\lambda & 4 \end{bmatrix} \; R_1 \leftrightarrows R_2$$

$$\sim \begin{bmatrix} 1 & 1-\lambda \\ 0 & 4-(1-\lambda)^2 \end{bmatrix} \; R_2 \to R_2 - (1-\lambda)\, R_1.$$

So the only way $Ax = 0$ will have a non trivial solution is if x_2 is a free variable which happens iff $4 - (1-\lambda)^2 = 0$, i.e. $4 = (1-\lambda)^2$ or $1 - \lambda = \pm 2$. **Ans:** $\lambda = -1$ **or** $\lambda = 3$. ■

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7. a. $T\left(\begin{bmatrix} 3\pi \\ 2\pi \end{bmatrix}\right) = \pi T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \pi \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\pi \\ 3\pi \end{bmatrix}.$

b. Since

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

the linearity of T implies;

$$T\begin{bmatrix} 0 \\ 2 \end{bmatrix} = T\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 3T\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}.$$

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8. Let $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ be three vectors in $\mathbb{R}^4$ and $A = [\mathbf{a}_1|\mathbf{a}_2|\mathbf{a}_3]$ – a 4×3 matrix. Answer the following questions true or false (no explanation need be given);

a: If there are real numbers x_1, x_2, and x_3 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = 0$, then $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a linearly dependent set. False

b: It is possible that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ span $\mathbb{R}^4$. False

c: It is possible that the equation $A\mathbf{x} = \mathbf{0}$ has no solutions. False

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9. Since

$$rref\,(A) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } x_4 \text{ is free,}$$

the solutions to the homogeneous equation $Ax = 0$ are of the form;

$$x = \begin{bmatrix} -2x_4 \\ x_4 \\ 0 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} : \; x_4 \text{ free}$$

and so the general solution to $A\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is of the form (particular + homogeneous solution)

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} : \; x_4 \text{ free.}$$

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10.

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

a.: Yes the columns of A span $\mathbb{R}^3$ since this will happen iff there is a pivot in every row of rref(A) which is true. Alternativle this happens iff $Ax = b$ has a solution for all $b \in \mathbb{R}^3$ and this holds as rref(A) has a pivot in every row.

b: No the columns of A are linearly dependent since rref(A) has a free variable, x_4. Taking $x_4 = 1$ shows in fact that

$$-2a_1 + a_2 + 0a_3 + a_4 = 0.$$

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