Midterm 2vS
Math 18, Section A
May 19, 2017
Time Limit: 50 Minutes

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- **DO NOT begin working, or even open this packet, until instructed to do so.**
- You should be in your assigned seat, unless instructed otherwise by Ed or one of the TAs.
- Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.
- You may use a two-sided page of notes on this exam.
- You may **not** use your books, additional notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** Unless otherwise directed in the statement of the problem, a correct answer, unsupported by calculations, explanation, or algebraic work will receive little or no credit. An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Question	Points
1	10
2	10
3	10
4	10
Total:	40

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1. Let $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ and let a, b, c, d be some numbers.

(a) (7 points) Compute A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{reorder}} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{scale}} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{shear}} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{shear}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \frac{3}{2} & 0 & -3 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 3/2 & 0 & -3 \\ 0 & 1/2 & 0 & 0 \\ 0 & -1/2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) (3 points) Solve the equation $A\vec{x} = (a, b, c, d)$. You may answer in terms of a, b, c and d.

$$\vec{x} = A^{-1} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a + \frac{3}{2}b - 3d \\ \frac{1}{2}b \\ -\frac{1}{2}b + d \\ c \end{bmatrix}$$

2. (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials with degree at most 2, and let $\mathcal{B} = \{1+x, 1+x^2, x+x^2\}$. $\mathcal{B}$ is a basis for $\mathbb{P}_2$ (and you don't need to prove this). Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy $T\left(\left[a_0 + a_1 x + a_2 x^2\right]_{\mathcal{B}}\right) = (a_1.a_2, a_3)$. Find the matrix corresponding to T.

T is the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C} = \{1, x, x^2\}$ so the matrix is

$$\begin{bmatrix} | & | & | \\ [1+x]_{\mathcal{C}} & [1+x^2]_{\mathcal{C}} & [x+x^2]_{\mathcal{C}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Alternate approach: ~~Call the matrix M.~~

$$T(\vec{e}_1) = T\left(\left[1+x\right]_{\mathcal{B}}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = T\left(\left[1+x^2\right]_{\mathcal{B}}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$T(\vec{e}_3) = T\left(\left[x+x^2\right]_{\mathcal{B}}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

So $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

3. (10 points) Let A be a matrix which is row-equivalent to $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and let $T : \mathbb{R}^3 \to \mathbb{R}^7$ be a linear transformation such that $T(\vec{x}) = \vec{0}$ if and only if $\vec{x} = A\vec{y}$ for some $\vec{y} \in \mathbb{R}^4$. What is the dimension of the range of T?

This says $\text{Nul } T = \text{col } A$

$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has two pivot positions, so

$\dim \text{col } A = 2$.

Rank theorem says

$$\dim \text{Range } T + \dim \text{Nul } T = 3$$

So $\dim \text{Range } T + 2 = 3$

So $\dim \text{Range } T = 1$

4. (a) (5 points) Find the volume of the parallelopiped in $\mathbb{R}^3$ with adjacent vertices $(0,1,0)$, $(0,1,2)$, $(1,1,1)$ and $(1,0,-1)$.

Subract $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ from each vertex, ~~so~~ to get a parallelpiped with the same volume and vertices

$\vec{0}$, $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

$$\text{Volume} = \left| \det \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 1 & -1 \end{bmatrix} \right| = \left| 2 \det \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right| = |-2| = 2$$

(b) (5 points) Let A and B be invertible matrices with $\det A = 4$ and $\det B = 2$. What is $\det(A^2B^{-1})$?

$\cancel{\det(A^2)\ de}$

$$\det(A^2B^{-1}) = \det(A)^2 \det(B)^{-1}$$
$$= \frac{4^2}{2} = 8$$