Final ExamvS
Math 18, Section A
June 14, 2017
Time Limit: 180 Minutes

Name (Print): ED

PID: ____________

- **DO NOT begin working, or even open this packet, until instructed to do so.**
- You should be in your assigned seat, unless instructed otherwise by Ed or one of the TAs.
- Enter all requested information on the top of this page, and put your name on the top of every page, in case the pages become separated.
- You may use a two-sided page of notes on this exam.
- You may **not** use your books, additional notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. Unless otherwise directed in the statement of the problem, a correct answer, unsupported by calculations, explanation, or algebraic work will receive little or no credit. An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Question	Points
1	10
2	10
3	10
4	10
5	14
6	6
7	10
8	10
Total:	80

DO NOT turn this page until instructed to do so

1. Let $A = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 3 \\ -1 & -2 & -3 & -4 \\ 2 & 4 & 6 & -3 \end{bmatrix}$. A is row-equivalent to $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) (4 points) Find a basis for the nullspace of A.

$$x_1 + 2x_2 + 3x_3 = 0 \qquad \Longrightarrow \qquad \vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
$$x_4 = 0$$

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) (2 points) Solve the equation $A\vec{x} = (2, 4, -2, 4)$. Express your answer in parametric vector form. Hint: the right hand side of that equation is one of the columns of A.

$$A\vec{e}_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \\ 4 \end{bmatrix}, \quad \text{so} \quad \vec{x} = \vec{e}_2 + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(c) (4 points) Find a basis for the column space of A.

Pivot columns are the 1^{st} and 4^{th}, so

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \\ 3 \end{bmatrix} \right\}$$ is a basis for col A

2. (10 points) Let V be the vector space of 3×3 matrices, $A \in V$ be an invertible matrix, and $H \subseteq V$ be the set of all matrices B such that AB is a diagonal matrix. It turns out that H is a subspace. Find $\dim H$.

We'll express H as a nullspace.

$$T: V \longrightarrow \mathbb{R}^6 \quad T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = \begin{bmatrix} \cancel{a_{11}} \\ a_{12} \\ a_{13} \\ a_{21} \\ \cancel{a_{22}} \\ a_{23} \\ a_{31} \\ a_{32} \\ \cancel{a_{33}} \end{bmatrix}$$

So M is diagonal if and only if $T(M) = \vec{0}$.

Let $Q: V \longrightarrow \mathbb{R}^3$ be $Q(B) = T(AB)$.

$H = \ker Q$. Q is onto:

$$Q\left(A^{-1}\begin{bmatrix} 7 & x_1 & x_2 \\ x_3 & 7 & x_4 \\ x_5 & x_6 & 7 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

So $\dim H = \dim \operatorname{Nul} Q$

$= \dim V - \dim \operatorname{range} Q$

$= 9 - 6$

$= 3$

3. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

(a) (5 points) Find A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 2 & 3 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{shear}} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{slear}} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 3 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{shear}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 6 & -1 & -1 & -2 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

(b) (5 points) Find $\det A$.

From the 2nd step above,

$$\det A = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (1)(1)(1)(1) = 1$$

5. Let $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$.

(a) (10 points) Find an eigenbasis for A.

$\det A - \lambda I = (3-\lambda)(-\lambda) + 2 = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$

$(A - 2I)\vec{x} = 0$

$\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \vec{x} = 0$

$\vec{x} = t\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$(A - 1I)\vec{x} = 0$

$\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \vec{x} = 0$

$\vec{x} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

(b) (4 points) Find an invertible matrix B such that $B^{-1}AB$ is a diagonal matrix.

Let $\mathfrak{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, so $[A]_{\mathfrak{B}}$ is diagonal.

$[A]_{\mathfrak{B}} = \underset{\mathfrak{B} \leftarrow \mathcal{E}}{P} \, A \, \underset{\mathcal{E} \leftarrow \mathfrak{B}}{P}$ $\qquad \mathcal{E}$ = standard basis

So set $B^{-1} = \underset{\mathfrak{B} \leftarrow \mathcal{E}}{P}$

$B = \underset{\mathcal{E} \leftarrow \mathfrak{B}}{P} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

4. (10 points) Let $\mathbb{P}$ be the vector space of polynomials in x and $H = \text{span}\{x, x^2\} \subset \mathbb{P}$. Let $\mathcal{B} = \{x, x^2 + x\}$; this is a basis for H. Let $T : H \to H$ be the linear transformation that sends $p(x)$ to $p(3x) + p(x)$. For example, $T(x^2 - x) = (3x)^2 - (3x) + x^2 - x = 10x^2 - 4x$. Find a matrix M satisfying

$$M[p(x)]_{\mathcal{B}} = [T(p(x))]_{\mathcal{B}}$$

$$M\vec{e}_1 = M[x]_{\mathcal{B}} = [T(x)]_{\mathcal{B}} = [3x + x]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$M\vec{e}_2 = [T(x^2 + x)]_{\mathcal{B}} = [9x^2 + 3x + x^2 + x]_{\mathcal{B}}$$

$$= [10x^2 + 4x]_{\mathcal{B}}$$

$10x^2 + 4x = 10(x^2 + x) - 6x$, so

$$= \begin{bmatrix} -6 \\ 10 \end{bmatrix}$$

$$M = \begin{bmatrix} 4 & -6 \\ 0 & 10 \end{bmatrix}$$

6. (a) (3 points) Find two 3×3 matrices A and B that satisfy all three of the following conditions:
   - The eigenvalues of A are 5 and 9.
   - The eigenvalues of B are 5 and 9 as well.
   - The characteristic polynomials of A and B are different.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix} \text{ and } \begin{bmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Char. polynomials are

$$(5-\lambda)^2(9-\lambda) \neq (5-\lambda)(9-\lambda)^2 \checkmark$$

(b) (3 points) Find a 3×3 matrix C that satisfies both of the following conditions:
   - The only eigenvalue of C is 4
   - The eigenspace of C with eigenvalue 4 is 2-dimensional.

$$C = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}. \text{ Then } C - 4I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ is}$$

rank-1, so the eigenspace has dimension

$3 - 1 = 2 \checkmark$

7. (10 points) Let

$$\vec{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

Find an **orthogonal** matrix A satifying both of the following conditions:

- $A\vec{e}_1 \in \text{span}\{\vec{b}_1\}$
- $A\vec{e}_2 \in \text{span}\{\vec{b}_1, \vec{b}_2\}$

Goal: Find $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ orthonormal, with

$\text{span}\{\vec{u}_1\} = \text{span}\{\vec{b}_1\}$

$\text{span}\{u_1, u_2\} = \text{span}\{b_1, b_2\}$

Then $A = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}$

$\vec{V}_1 = \vec{b}_1$

$$\vec{V}_2 = \vec{b}_2 - \frac{\vec{b}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{V}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is orthogonal to $\vec{v}_1, \vec{v}_2$

So $\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad u_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & \sqrt{1/2} & \sqrt{1/2} \\ 1 & 0 & -\sqrt{1/2} \\ 0 & \sqrt{1/2} & 0 \end{bmatrix}$$

8. (10 points) Let

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}, \qquad \vec{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Let A be some 3×2 matrix with linearly independant columns $\vec{b}_1, \vec{b}_2$, so $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ is a basis for $\operatorname{Col} A$, and suppose $A = QR$ is a QR-decomposition of A. Let $\hat{y}$ be the orthogonal projection of $\vec{y}$ onto $\operatorname{Col} A$. Find the coordinate vector $[\hat{y}]_{\mathcal{B}}$. You may answer in terms of R.

$$\hat{y} = \left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{1/3} \\ \sqrt{1/3} \\ \sqrt{1/3} \end{bmatrix} \right) \begin{bmatrix} \sqrt{1/3} \\ \sqrt{1/3} \\ \sqrt{1/3} \end{bmatrix} + \left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \sqrt{1/2} \\ -\sqrt{1/2} \end{bmatrix} \right) \begin{bmatrix} 0 \\ \sqrt{1/2} \\ -\sqrt{1/2} \end{bmatrix}$$

$$= \frac{6}{\sqrt{3}} \begin{bmatrix} \sqrt{1/3} \\ \sqrt{1/3} \\ \sqrt{1/3} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \sqrt{1/2} \\ -\sqrt{1/2} \end{bmatrix}, \quad \text{So if } \mathcal{U} = \left\{ \begin{bmatrix} \sqrt{1/3} \\ \sqrt{1/3} \\ \sqrt{1/3} \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{1/2} \\ -\sqrt{1/2} \end{bmatrix} \right\},$$

$$[\hat{y}]_{\mathcal{U}} = \begin{bmatrix} 6/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}$$

$$R = \underset{\mathcal{U} \leftarrow \mathcal{B}}{P}, \quad \text{so}$$

$$[\hat{y}]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{U}}{P} \begin{bmatrix} 6/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix} = \underset{\mathcal{U} \leftarrow \mathcal{B}}{P}^{-1} \begin{bmatrix} 6/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= R^{-1} \begin{bmatrix} 6/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}$$