Solutions

Name: ______________________________

Student ID No.: ______________________________

Discussion Section: ______________________________

Math 20F Final Exam(ver. c)

Winter 2016

Problem	Score
1	/48
2	/26
3	/27
4	/24
5	/24
6	/24
7	/27
Total	/200

1. (48 Points.) The following are True/False questions. **For this problem only, you do not have to show any work.** There will be no partial credit given for this problem. For this problem:

- A correct answer gives 4 points.
- An incorrect answer gives 0 points.
- If you leave the space blank, you receive 2 points.

__T__ (a) If a vector space V is spanned by p vectors, then it's impossible for a set of more than p vectors to be linearly independent.

__T__ (b) If the LU-factorization of an $n \times n$ matrix A has a zero in either the diagonal of L or U, then A is not invertible.

__T__ (c) Assume the vector space V has an inner product and that $\vec{u}, \vec{y} \in V$. The two orthogonal projections: $\text{proj}_{\vec{u}}\vec{y}$ and $\text{proj}_{3\vec{u}}\vec{y}$ must be equal.

__F__ (d) Assume A is an $m \times n$ matrix and $\vec{b}$ is a vector in $\mathbf{R^m}$. Then the set of solutions to $A\vec{x} = \vec{b}$ is a subspace of $\mathbf{R^n}$.

__T__ (e) If the homogeneous problem $A\vec{v} = \vec{0}$ only has the trivial solution, then whenever $A\vec{x} = \vec{b}$ has a solution, it is unique.

__F__ (f) The product of two nonzero matrices must be a nonzero matrix.

__T__ (g) If the columns of an $n \times n$ matrix are linearly independent, then it is invertible.

__F__ (h) If two matrices are similar, then any eigenvector of one of the matrices is also an eigenvector of the other.

__T__ (i) If $T : V \to W$ is a linear transformation, then $T(\vec{0}) = \vec{0}$.

__T__ (j) Consider the vector space V consisting of continuous functions defined on the interval $[0,1]$. The subset consisting of functions f satisfying $\int_0^1 f\,dx = 0$, is a subspace of V.

__F__ (k) If a system of equations is inconsistent, then the corresponding homogeneous system must have nontrivial solutions.

__T__ (l) If, at a critical point of the multivariable function $f(x,y)$, the eigenvalues of the matrix $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ are both positive, then f has a local minimum at that critical point.

2. (26 Points.) Let A be the matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 2 & 4 & -2 & 4 & 6 \\ 1 & 2 & -1 & 3 & 1 \\ 1 & 2 & 1 & 4 & 6 \end{bmatrix}$$

Think of A as a linear transformation $T : \mathbf{R^5} \to \mathbf{R^4}$ defined by $T(\vec{x}) = A\vec{x}$.

(a) Find a basis for the Range of T.

(b) Find a basis for the Null Space of T.

(c) Define the linear transformation $L : \mathbf{R^4} \to \mathbf{R^5}$ by $L(\vec{x}) = A^T\vec{x}$. Find a basis for the Range of L.

Row reduction gives:

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 0 & -2 & -1 & -5 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) *There are pivots in columns 1, 3, and 4 so the first, third, and fourth columns form a basis for the Range of T:*

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \\ 4 \end{bmatrix} \right\}.$$

(b) *Solving the homogeneous problem using the row reduction above gives solutions of the form:*

$$\begin{bmatrix} -2x_2 - \frac{21}{2}x_5 \\ x_2 \\ -\frac{7}{2}x_5 \\ 2x_5 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{21}{2} \\ 0 \\ -\frac{7}{2} \\ 2 \\ 1 \end{bmatrix}.$$

So

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{21}{2} \\ 0 \\ -\frac{7}{2} \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the null space of T.

(c) *The entries of the nonzero rows of the matrix above form a basis for the Range of L:*

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \\ -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 4 \end{bmatrix} \right\}.$$

3. The following are all eigenvalue/eigenvector problems.

(a) (9 Points.) Find all the eigenvalues of the following matrix. Also, find a basis for each eigenspace. Is this matrix diagonalizable?

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & -5 & -15 \\ 0 & 2 & 6 \end{bmatrix}.$$

(b) (9 Points.) Find all the eigenvalues of the following matrix. Also, find a basis for each eigenspace. Is this matrix diagonalizable?

$$B = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}.$$

(c) (9 Points.) Let P_3 be the vector space of polynomials of degree 3 or less. Find all the eigenvalues of the linear transformation $T : P_3 \to P_3$ defined by $T(p(x)) = p''(x)$. Also, find a basis for each eigenspace.

(a) *The characteristic polynomial is:* $-\lambda^3 + 2\lambda^2 + \lambda = -\lambda(1-\lambda)^2$, *so the eigenvalues are* 0 *and* 1. *Solving* $A\vec{x} = \vec{0}$ *gives the eigenspace corresponding to* $\lambda = 0$ *and its basis:*

$$\left\{ \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

Solving $(A\vec{-}I)x = \vec{0}$ *gives the eigenspace corresponding to* $\lambda = 1$ *and its basis:*

$$\left\{ \begin{bmatrix} 0 \\ -\frac{5}{2} \\ 1 \end{bmatrix} \right\}.$$

Because the eigenvalue $\lambda = 1$ *has multiplicity 2 and eigenspace only dimension 1, there cannot be a basis of* $\mathbf{R^3}$ *consisting of eigenvectors, and so this matrix is not diagonalizable.*

(b) *The characteristic polynomial is:* $-\lambda^3 + 4\lambda^2 - 4\lambda = -\lambda(2-\lambda)^2$, *so the eigenvalues are* 0 *and* 2. *Solving* $A\vec{x} = \vec{0}$ *gives the eigenspace corresponding to* $\lambda = 0$ *and its basis:*

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Solving $(A - 2I)\vec{x} = \vec{0}$ *gives the eigenspace corresponding to* $\lambda = 2$ *and its basis:*

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This matrix is diagonalizable because putting together our basis vectors for each eigenspace gives us enough vectors to form a basis for $\mathbf{R^3}$.

(c) *There are a couple ways to do this one. You can convert the entire problem into a matrix problem by choosing a basis for* $\mathbf{P_3}$, *for example. However, let's do this one more directly:* λ *is an eigenvalue of* T *means* $T(p(t)) = \lambda p(t)$. *If we write* $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, *then finding eigenvalues/eigenvectors is the same as solving:*

$$2a_2 + 6a_3 t = p''(t) = T(p(t)) = \lambda p(t) = \lambda(a_0 + a_1 t + a_2 t^2 + a_3 t^3).$$

Matching terms, we get:

$$\begin{aligned}
2a_2 &= \lambda a_0, \\
6a_3 &= \lambda a_1, \\
0 &= \lambda a_2, \\
0 &= \lambda a_3,
\end{aligned}$$

For $\lambda = 0$*, the solutions to this* $a_2 = a_3 = 0$ *while* a_0 *and* a_1 *are free. This tells us the eigenspace corresponding to* $\lambda = 0$ *is the set of polynomials of the form* $a_0 + a_1 t$*, which is spanned by* $\{1, t\}$*.*

And if $\lambda \neq 0$*, the last two equations say that* $a_3 = a_2 = 0$*, which, plugged into the first two, gives* $a_1 = a_0 = 0$*. So there are no nonzero eigenvalues.*

4. The following problems are both computational problems. They are otherwise unrelated.

(a) (12 Points.) Suppose an economy has two sectors, Goods and Services. Each year, Goods sells 80% of its output to Services and keeps the rest, while Services sells 70% of its output to Goods and retains the rest. Find equilibrium prices for the annual outputs of the Goods and Services sectors that make each sector's income match its expenditures.

(b) (12 Points.) Consider the vector space, $\mathbf{P_2}$, of all polynomials of degree 2 of less. Let $T : \mathbf{P_2} \to \mathbf{P_2}$ be the (invertible) linear tranformation defined by $T(1) = 1 + 2t + t^2$, $T(t) = 2 - t + 3t^2$, and $T(t^2) = t - t^2$.

Find $T^{-1}(1)$, $T^{-1}(t)$, and $T^{-1}(t^2)$.

(a) *If we represent Goods with the first coordinate entry of* $\mathbf{R^2}$ *and Services with the second, then finding equilibrium prices is equivalent to finding an eigenvector corresponding to* $\lambda = 1$ *for the matrix:*

$$\begin{bmatrix} \frac{1}{5} & \frac{7}{10} \\ \frac{4}{5} & \frac{3}{10} \end{bmatrix}.$$

One solution is when Goods is at $\frac{7}{8}$ *and Services is at* 1.

(b) *Converting this into a matrix problem is one way to do this. Use* $\{1, t, t^2\}$ *as a basis for* $\mathbf{P_2}$. *Then the linear transformation T becomes:*

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

Invert this matrix to get:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{7}{4} & -\frac{1}{4} & -\frac{5}{4} \end{bmatrix}.$$

Translating back to polynomials, we have:

$$\begin{aligned} T^{-1}(1) &= -\tfrac{1}{2} + \tfrac{3}{4}t + \tfrac{7}{4}t^2, \\ T^{-1}(t) &= \tfrac{1}{2} - \tfrac{1}{4}t - \tfrac{1}{4}t^2, \\ T^{-1}(t^2) &= \tfrac{1}{2} - \tfrac{1}{4}t - \tfrac{5}{4}t^2. \end{aligned}$$

5. The following problems are both computational problems. They are otherwise unrelated.

(a) (12 Points.) Consider the following vectors in $\mathbf{R^4}$:

$$\vec{b_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{b_2} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \quad \vec{b_3} = \begin{bmatrix} 6 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad \vec{b_4} = \begin{bmatrix} 14 \\ -66 \\ 41 \\ 9 \end{bmatrix}, \quad \text{and} \quad \vec{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}.$$

The set $\left\{\vec{b_1}, \vec{b_2}, \vec{b_3}, \vec{b_4}\right\}$ forms an orthogonal basis for $\mathbf{R^4}$. That means $\vec{v}$ can be written as a linear combination: $\vec{v} = c_1\vec{b_1} + c_2\vec{b_2} + c_3\vec{b_3} + c_4\vec{b_4}$. Find c_2 and c_3.

(b) (12 Points.) For the vector space $\mathbf{R^2}$, find the change-of-coordinates matrix from the basis $\left\{ \begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ -5 \end{bmatrix} \right\}$ to the basis $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

(a) *Because we have an orthogonal basis,*

$$\begin{aligned} c_2 &= \frac{\vec{v}\cdot\vec{b_2}}{\|\vec{b_2}\|} = \frac{3}{14}, \\ c_3 &= \frac{\vec{v}\cdot\vec{b_3}}{\|\vec{b_3}\|} = \frac{5}{41}. \end{aligned}$$

(b) *The columns of the matrix that we want are formed by writing the vectors of the first basis in terms of the second basis. Doing this (which is equivalent to solving 2×2 systems) gives:*

$$\begin{bmatrix} -1 \\ 8 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 4 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \textit{and} \quad \begin{bmatrix} -1 \\ 5 \end{bmatrix} = -\frac{4}{3}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So the change of basis matrix is:

$$\begin{bmatrix} 3 & -\frac{4}{3} \\ -4 & \frac{1}{3} \end{bmatrix}.$$

6. The following are all proof-type problems. They are otherwise unrelated.

(a) (8 Points.) Suppose the solutions of a homogeneous system of five linear equations in six unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations? Justify your answer.

(b) (8 Points.) Let A be an $n \times n$ matrix that has at least one eigenvalue. Is the set of all eigenvectors of A necessarily a subspace of $\mathbf{R}^{\mathbf{n}}$? If so, prove it. Otherwise provide a counterexample.

(c) (8 Points.) Consider the vector space C of continuous functions defined on all of $\mathbf{R}$. Show that

$$f \cdot g = \int_0^1 f(x)g(x)\,dx$$

is *not* an inner product. **Note:** *You'll probably want to work with functions that have certain properties. If you cannot come up with the formulas for such functions, you may draw their graphs instead.*

(a) *Representing the homogeneous system of equations in matrix form: $A\vec{x} = \vec{0}$, the fact that all solutions are multiples of one nonzero solution means that the nullspace of A is of dimension one. But $\mathrm{rank}A + \dim \mathrm{Null}A = 6$, so the rank of A must be five, which means A – as a mapping – must be onto, and so $A\vec{x} = \vec{b}$ must be always be solvable for every $\vec{b}$.*

(b) *The set of all eigenvectors is not necessarily a subspace (Note that the set of all eigenvectors corresponding to the same eigenvalue is a subspace, however). Any example of a matrix with two distinct eigenvalues will give a counterexample. For example, consider the matrix:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors (corresponding to $\lambda = 1$ and $\lambda = 2$). $A(\vec{e}_1 + \vec{e}_2) = \vec{e}_1 + 2\vec{e}_2$ is clearly not a constant multiple of $\vec{e}_1 + \vec{e}_2$ and is thus not an eigenvector of A. Because the sum of these two eigenvectors is not itself an eigenvector, the set of eigenvectors cannot be a subspace of $\mathbf{R}^2$.

(c) *This one was supposed to be hard. An inner product must satisfy the property that only the zero vector has length zero (where the length is, by definition square root of the vector dotted with itself). However, any continuous function that is zero in the interval $[0,1]$ and has nonzero values somewhere outside that interval will violate this property (because this proposed inner product only integrates between 0 and 1). For example, this function will work:*

$$f(x) = \begin{cases} -x & x < 0 \\ 0 & x \geq 0 \end{cases}$$

This is why every time we discussed inner products on function spaces, we were careful to say that our vector space was not all of C, but rather $C[0,1]$ – the space of functions defined on the interval $[0,1]$.

Exercise. *Show that for the vector space P of polynomials,*

$$f \cdot g = \int_0^1 f(x)g(x)\,dx$$

is indeed an inner product. Why does this inner product work for polynomials and not the space of continuous functions?

7. The following are all proof-type problems. They are otherwise unrelated.

(a) (9 Points.) Assume A is a 7×4 matrix and B is a 4×7 matrix. Show that AB is not invertible.

(b) (9 Points.) Assume $T : V \to W$ is a linear transformation between vector spaces V and W. Show that if $\{\vec{v}_1, \dots \vec{v}_p\}$ are vectors in V such that $\{T(\vec{v}_1), \dots T(\vec{v}_p)\}$ are linearly independent, then $\{\vec{v}_1, \dots \vec{v}_p\}$ is also linearly independent.

For full credit, mark with a star (*) everywhere you use the fact that T is a linear transformation.

(c) (9 Points.) Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Say that Z is a subspace of W. Let U be the set of all vectors $\vec{x} \in V$ such that $T(\vec{x}) \in Z$. Show that U is a subspace of V.

For full credit, mark with a star (*) everywhere you use the fact that T is a linear transformation. Also, mark with two stars (**) everywhere you use the fact that Z is a subspace of W.

(a) *Same problem as on Midterm 2.*

(b) *Assume we have $c_1\vec{v}_1 + \cdots c_p\vec{v}_p = \vec{0}$. We must show that the only way this can be true is if: $c_1 = c_2 = \cdots = c_p = 0$. Apply T to both sides and using the fact that T is a linear transformation (twice):*

$$c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p) \overset{(*)}{=} T(c_1\vec{v}_1) + \cdots + T(c_p\vec{v}_p) \overset{(*)}{=} T(c_1\vec{v}_1 + \cdots c_p\vec{v}_p) = T(\vec{0}) \overset{(*)}{=} \vec{0}.$$

(The last equality uses the fact that T is a linear transformation indirectly.) So we conclude $c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p) = \vec{0}$, but because the $T(\vec{v}_1), \dots T(\vec{v}_p)$ are linearly independent, that implies $c_1 = c_2 = \cdots = c_p = 0$, which is what we wanted.

(c) *We must show that (1), if $\vec{x}, \vec{y} \in U$ then $\vec{x} + \vec{y} \in U$, and (2), if c is a scalar and $\vec{x} \in U$, then $c\vec{x} \in U$.*

For (1), to show, $\vec{x} + \vec{y} \in U$ means we must show $T(\vec{x} + \vec{y}) \in Z$. Because T is a linear transformation (*), that's the same as showing $T(\vec{x}) + T(\vec{y}) \in Z$. But that's true because $x \in U$ and $y \in U$ mean precisely that $T(\vec{x}) \in Z$ and $T(\vec{y}) \in Z$, so since Z is a subspace of W (**), we have $T(\vec{x}) + T(\vec{y}) \in Z$, which is what we wanted.

For (2), to show, $c\vec{x} \in U$ means we must show $T(c\vec{x}) \in Z$. Because T is a linear transformation (*), that's the same as showing $cT(\vec{x}) \in Z$. But that's true because $x \in U$ means precisely that $T(\vec{x}) \in Z$, so since Z is a subspace of W (**), we have $cT(\vec{x}) \in Z$, which is what we wanted.

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