Solutions

Name: ______________________________

Student ID No.: ______________________________

Discussion Section: ______________________________

Math 20F Midterm I (ver. a)

Winter 2016

Problem	Score
1	/24
2	/24
3	/26
4	/26
Total	/100

1. (24 Points.) The following are True/False questions. **For this problem only, you do not have to show any work.** There will be no partial credit given for this problem. For this problem:

- A correct answer gives 4 points.
- An incorrect answer gives 0 points.
- If you leave the space blank, you receive 2 points.

__F__ (a) If $T : \mathbf{R^n} \to \mathbf{R^m}$ and $T(\vec{0}) = \vec{0}$, then T is a linear transformation.

__T__ (b) Assume A is an $m \times n$ matrix. If $A\vec{x} = \vec{b}$ has a solution for every vector $\vec{b}$ then the columns of A span $\mathbf{R^m}$.

__T__ (c) If the Row Echelon form of A has a pivot in every row then $A\vec{x} = \vec{b}$ has a solution for every vector $\vec{b}$.

__F__ (d) Whenever $\vec{x}_1$ and $\vec{x}_2$ are two solutions to $A\vec{x} = \vec{b}$, then $\vec{x}_1 + \vec{x}_2$ is also a solution to $A\vec{x} = \vec{b}$.

__T__ (e) If the homogeneous problem: $A\vec{x} = \vec{0}$ has only the trivial solution, then the problem $A\vec{x} = \vec{b}$ either has no solutions or has exactly one solution.

__T__ (f) If the Row Echelon form of the matrix A has a column without a pivot, then the problem $A\vec{x} = \vec{0}$ has many solutions.

2. Each statement below is either true (in all cases) or false (in at least one case). If false, construct a counterexample. If true, give a justification using the definition of "linearly independent."

(a) (8 Points.) If $v_1, \ldots, v_4$ are in $\mathbf{R^5}$ and $\{v_1, v_2, v_3\}$ is linearly dependent then $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

(b) (8 Points.) If $v_1, \ldots, v_4$ are in $\mathbf{R^5}$ and v_1 is *not* a linear combination of $\{v_2, v_3, v_4\}$, then $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

(c) (8 Points.) If $v_1, \ldots, v_4$ are linearly independent vectors in $\mathbf{R^5}$, then $\{v_1, v_2, v_3\}$ is also linearly independent.

(a) *True.* $\{v_1, v_2, v_3\}$ *linearly dependent means there exists weights* a_1, a_2, *and* a_3 *not all zero such that*

$$a_1v_1 + a_2v_2 + a_3v_3 = 0.$$

This means

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0,$$

for $a_4 = 0$ *and the same* a_1, a_2, a_3 *as above. Since not all of* a_1, a_2, *and* a_3 *are zero, that means not all of of* a_1, a_2, a_3, a_4 *are zero. And that means* $\{v_1, v_2, v_3, v_4\}$ *are linearly dependent.*

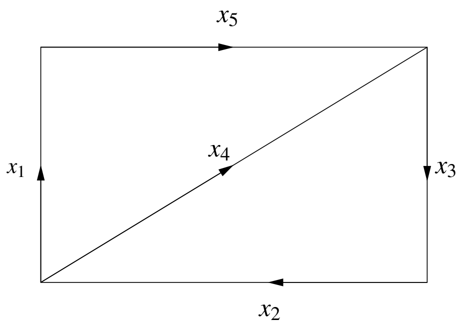
(b) *False. Here's a simple counterexample: choose* $v_1 = e_1$ *and* $v_2 = v_3 = v_4 = e_2$. *Then* v_1 *is not a linear combination of* $\{v_1, v_2, v_3\}$, *yet* $\{v_1, v_2, v_3, v_4\}$ *are linearly dependent (because, for example,* $0v_1 + 0v_2 + v_3 - v_4 = 0$*).*

(c) *True. To show that* $\{v_1, v_2, v_3\}$ *is linearly independent, we must show that if* $a_1v_1 + a_2v_2 + a_3v_3 = 0$, *then* $a_1 = a_2 = a_3 = 0$ *("only the trivial solution...").*

So assume $a_1v_1 + a_2v_2 + a_3v_3 = 0$. *Then* $a_1v_1 + a_2v_2 + a_3v_3 + 0v_4 = 0$ *must also be true (we simply added* 0 *to the left hand side). But because* $\{v_1, v_2, v_3, v_4\}$ *are linearly independent, that means all the weights in this expression must be zero.*

In particular, $a_1 = a_2 = a_3 = 0$. *Which is what we wanted.*

3. (25 Points.) Find the general flow pattern of the network shown in the figure.



The flow into each corner must equal the flow out. This gives us the system of equations:

$$
\begin{aligned}
x_1 + \quad &= x_5, \\
x_4 + x_5 &= x_3, \\
x_3 &= x_2, \\
x_2 &= x_1 + x_4,
\end{aligned}
$$

which is equivalent to the homogeneous system:

$$
\begin{aligned}
x_1 - x_5 &= 0, \\
-x_3 + x_4 + x_5 &= 0, \\
-x_2 + x_3 &= 0, \\
-x_1 + x_2 - x_4 &= 0.
\end{aligned}
$$

Solving this system gives $x_1 = x_5$, $x_2 = x_4 + x_5$, $x_3 = x_4 + x_5$, $x_4 = \text{free}$, *and* $x_5 = \text{free}$.

4. Each of the following two parts are about linear transformations (but are otherwise not related).

(a) (13 Points.) Consider the transformation $T : \mathbf{R^3} \to \mathbf{R^2}$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1x_2 \\ x_3 \end{bmatrix}.$$

Is T a linear transformation? You must justify your answer using the definition of linear transformation.

(b) (13 Points.) Assume $U : \mathbf{R^2} \to \mathbf{R^3}$ is a linear transformation where

$$U\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \quad \text{and} \quad U\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

Find $U\left(\begin{bmatrix} 14 \\ 4 \end{bmatrix}\right)$.

(a) *T is not a linear transformation. If it were, it would have to satisfy:*

$$T\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = cT\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right).$$

But the left hand side is equal to $\begin{bmatrix} c^2x_1x_2 \\ cx_3 \end{bmatrix}$*, while the left hand side is equal to* $c\begin{bmatrix} x_1x_2 \\ x_3 \end{bmatrix}$*. For these two vectors to be equal, we'd need* $cx_1x_2 = c^2x_1x_2$ *for all possible choices of* c, x_1, *and* x_2*. Clearly this isn't true whenever* $c \neq 1$ *and* x_1 *and* x_2 *not both zero: for example, with* $c = 2$, $x_1 = 1$, *and* $x_2 = 1$ *(*x_3 *can be anything), we would not have equality. In short, we have*

$$T\left(2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix},$$

while:

$$2T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

(Note that $T(u+v) = T(u) + T(v)$ *is generally not true either.)*

(b) *We first write* $\begin{bmatrix} 14 \\ 4 \end{bmatrix}$ *as a linear combination of* $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$*. This involves solving the system of equations*

$$\begin{bmatrix} 14 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The solution is:

$$\begin{bmatrix} 14 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Therefore, because U is a linear transformation, *we have:*

$$U\left(\begin{bmatrix} 14 \\ 4 \end{bmatrix}\right) = U\left(4\begin{bmatrix} 3 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 4U\left(\begin{bmatrix} 3 \\ 0 \end{bmatrix}\right) + 2U\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 4\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 16 \\ 2 \end{bmatrix}.$$