Solutions

Name: ______________________________

Student ID No.: ______________________________

Discussion Section: ______________________________

Math 20F Midterm II(ver. a)

Winter 2016

Problem	Score
1	/24
2	/25
3	/27
4	/24
Total	/100

1. (24 Points.) The following are True/False questions. **For this problem only, you do not have to show any work.** There will be no partial credit given for this problem. For this problem:

- A correct answer gives 4 points.
- An incorrect answer gives 0 points.
- If you leave the space blank, you receive 2 points.

__T__ (a) If a vector space V has p linearly independent vectors, then it's impossible for a set of fewer than p vectors to span V.

__F__ (b) If A is an $m \times n$ matrix whose columns are linearly dependent, then the column space of A must have dimension less than m.

__F__ (c) Let W be the subset of $\mathbf{R}^2$ consisting of all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $xy \leq 0$. W is a subspace of $\mathbf{R}^2$.

__T__ (d) Let A be an $n \times n$ matrix. If there is a $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solutions, then A cannot be one-to-one.

__T__ (e) Let A be a matrix. Any column of A that does not have a pivot (when A is put into Row Echelon form) can be written as a linear combination of the columns of A that do have pivots.

__F__ (f) Let $C(\mathbf{R})$ be the vector space of all continuous functions defined on the real line. Let H be the subset consisting of all continuous functions $f(x)$ such that $f(3) \geq 0$. H is a subspace of $C(\mathbf{R})$.

2. (25 Points.) Given below is the LU-factorization of a matrix A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \\ 0 & 3 & -7 \\ 0 & 0 & 5 \end{bmatrix}.$$

Solve $A\vec{x} = \vec{b}$, where

$$\vec{b} = \begin{bmatrix} 9 \\ 25 \\ -10 \end{bmatrix}.$$

In order to receive credit, you must use the LU-factorization to solve this problem.

First solve:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 25 \\ -10 \end{bmatrix}.$$

The solution is:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 20 \end{bmatrix}.$$

Plug this into:

$$\begin{bmatrix} 4 & 3 & 2 \\ 0 & 3 & -7 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

and solve again for the final answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{17}{2} \\ \frac{35}{3} \\ 4 \end{bmatrix}.$$

3. The following are all proof-type problems. They are otherwise unrelated.

(a) (9 Points.) Let T be a linear transformation from a vector space V to a vector space W. Show that the kernel of T is a subspace of V. *You must note every place where you use the fact that T is a linear transformation in order to receive full credit.*

(b) (9 Points.) Show that the vector space $C(\mathbf{R})$ of all continuous functions defined on the real line is infinite-dimensional. *Write down the statements of any Theorems you use.*

(c) (9 Points.) Assume A is a 6×4 matrix and B is a 4×6 matrix. Show that AB is not invertible.

(a) ***(1)*** *Whenever $\vec{u}$ and $\vec{v}$ are in the kernel of T, we have*

$$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) = \vec{0} + \vec{0} = \vec{0}.$$

So $\vec{u}+\vec{v}$ is also in the kernel of T ("the kernel is closed under addition"). Note that in the first equality we used the fact that T is a linear transformation.

(2) *If $c \in \mathbf{R}$ and $\vec{u}$ is in the kernel of T, then*

$$T(c\vec{u}) = cT(\vec{u}) = c\vec{0} = \vec{0}.$$

So $c\vec{u}$ is in the kernel of T ("the kernel is closed under scalar multiplication"). Note that in the first equality we used the fact that T is a linear transformation.

(1) and (2) together tells us the kernel of T is a subspace of V.

(b) *You can write this one up in many different ways, but the general idea is that the vector space of polynomials is infinite dimensional and is a subspace of $C(\mathbf{R})$, therefore $C(\mathbf{R})$ must be infinite dimensional as well (a subspace can't be "bigger" than the vector space it lives in).*

Here's one way to write it up:

If $C(\mathbf{R})$ were finite dimensional (of dimension n), then any set of vectors with more than n vectors must be linearly dependent. ("If V is a finite dimensional vector space, then any set of more than $\dim V$ vectors must be linearly dependent"). However, $\{1, t, t^2, \ldots, t^n\}$ is a linearly independent set in $C(\mathbf{R})$ of $n+1$ elements. Therefore $C(\mathbf{R})$ can't be finite dimensional.

How do we know the above set is linearly independent? Assume

$$a_0 + a_1 t + \cdots + a_n t^n = 0.$$

For our set to be linearly independent, we need to show each $a_i = 0$. Take the biggest k such that $a_k \neq 0$ (if every $a_i = 0$, then we're done). Divide through by $a_k t^k$, giving us:

$$\frac{a_0}{a_k}\frac{1}{t^k} + \frac{a_1}{a_k}\frac{1}{t^{k-1}} + \cdots + \frac{a_{k-1}}{a_k}\frac{1}{t} + 1 = 0.$$

Choose t so large that every term in the sum above (besides the last term) has absolute value less than $\frac{1}{k}$ (think about why this is possible). Then for that t, the left hand side can't be equal to zero. So it must be that all the coefficients a_i are equal to zero.

Note that you did not have to be so detailed on the exam.

(c) *There are many ways to do this one. Here's one:*

Think of AB as the composition of two linear transformations: $B : \mathbf{R}^6 \to \mathbf{R}^4$, followed by $A : \mathbf{R}^4 \to \mathbf{R}^6$. With this point of view, it's easy to see that if B is not one-to-one, then the composition can't be either. So the problem boils down to showing that B is not one-to-one. But B has more columns than rows so it can't be one-to-one.

Here it is in more detail:

We'll show AB is not one-to-one. Because B is a linear transformation from a "bigger" vector space to a "smaller" one, it can't be one-to-one: More precisely, you can say that its Row Echelon form must have at least 2 columns without pivots so there are many solutions to $B\vec{x} = \vec{0}$, so it's not one-to-one. So there are two different vectors $\vec{v}_1$ and $\vec{v}_2$ such that $B\vec{v}_1 = B\vec{v}_2$. Applying A to both sides, we get $AB\vec{v}_1 = AB\vec{v}_2$. Because $\vec{v}_1 \neq \vec{v}_2$, AB is not one-to-one, so it can't be invertible.

Note that you can also do this problem by showing AB is not onto, which boils down to showing A is not onto, which boils down to the fact that A has more rows than columns.

4. The following problems are computational. They are otherwise unrelated.

(a) (12 Points.) Let $T : \mathbf{R^2} \to \mathbf{R^2}$ be a linear transformation that first reflects through the x_1-axis, then rotates $\frac{\pi}{4}$ radians counterclockwise. Find the standard matrix of this linear transformation.

(b) (12 Points.) Determine whether the polynomials $1+2x+x^2+x^3$, $-1+8x-3x^2+x^3$, and $2-x+3x^2+x^3$. are linearly independent (in the vector space $\mathbf{P_3}$).

(a) *$\vec{e}_1$ first goes (under reflection) to itself, and then to* $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$. *$\vec{e}_2$ first goes (under reflection) to $-\vec{e}_2$, and then to* $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$. *So the standard matrix is:*

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

(a) *This is most easily done using coordinates. Using the basis $\{1, x, x^2, x^3\}$ we know that our polynomials are linearly independent if and only if the vectors:*

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

are linearly independent. Putting these into a matrix and row reducing, we get:

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 8 & -1 \\ 1 & -3 & 3 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 10 & -5 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 10 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is a column without a pivot, so the homogeneous problem has many solutions, so the vectors (and hence the polynomials) are not linearly independent.