**Math 18D** **Final Exam ver. A** **Name:** ______________

**March 24, 2017** **(Total Points: *50*)** **PID:** ______________

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**Instructions**

1. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
2. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
3. **Read each question carefully and answer each question completely.**
4. **Show all of your work. No credit will be given for unsupported answers, even if correct.**
5. **Write your Name at the top of each page.**

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(1 point) 1. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 2. Determine if the following system of linear equations is consistent or not. If the system is consistent, describe the solution set by using parametric form.

$$\begin{cases} x_1 \quad\quad + x_3 + x_4 - 3x_5 = -2 \\ -x_1 + x_2 - 10x_3 - x_4 + 4x_5 = 7 \\ \quad -4x_2 + 36x_3 + x_4 - 3x_5 = -21 \end{cases}.$$

Sol:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -3 & -2 \\ -1 & 1 & -10 & -1 & 4 & 7 \\ 0 & -4 & 36 & 1 & -3 & -21 \end{bmatrix} \sim_{\text{pivot at (1,1)}}^{r_2 \to r_2 + r_1} \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & -2 \\ 0 & 1 & -9 & 0 & 1 & 5 \\ 0 & -4 & 36 & 1 & -3 & -21 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_3 \to r_3 + 4r_2} \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & -2 \\ 0 & 1 & -9 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

$$\sim_{\text{pivot at (3,4)}}^{r_1 \to r_1 - r_3} \begin{bmatrix} 1 & 0 & 1 & 0 & -4 & -1 \\ 0 & 1 & -9 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

The general solution of the linear system is: $\begin{bmatrix} -1 \\ 5 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 14 \\ 1 \\ -1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 4 \\ 0 \\ -2 \\ 1 \end{bmatrix}$, where $x_3$ and $x_5$ are free.

3. Given that $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_6] = \begin{bmatrix} 1 & 1 & 0 & 3 & -2 & 3 \\ -4 & -5 & 0 & -12 & 9 & -13 \\ -1 & 4 & -1 & 4 & -2 & -7 \\ -2 & 3 & -1 & 1 & 1 & -12 \end{bmatrix}$ is row equivalent to $B = [\mathbf{b}_1\ \mathbf{b}_2 \cdots \mathbf{b}_6] = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -7 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$. Let $H$ be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $K$ be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}$. Find a basis for each of the following subspaces.

(3 points) (a) $\text{Row}(A)^\perp$, the orthogonal complement of the row space of $A$.

(2 points) (b) $H + K$, the sum of the two subspaces $H$ and $K$.

(2 points) (c) $H \bigcap K$, the intersection of the two subspaces $H$ and $K$.

Sol: (a) The basis of $\text{Row}(A)^\perp$ is $\{ \begin{bmatrix} 0 \\ 1 \\ -7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} \}$

(b) The basis of $H + K$ is $\{ \begin{bmatrix} 1 \\ -4 \\ -1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 4 \\ 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} -2 \\ 9 \\ -2 \\ 1 \end{bmatrix} \}$

(c) The basis of $H \bigcap K$ is $\{\mathbf{a}_4, 2\mathbf{a}_5 + \mathbf{a}_6\}$, or $\{ \begin{bmatrix} 3 \\ -12 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ -11 \\ -10 \end{bmatrix} \}$

4. Let $T$ be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T(a_0, a_1) = (2\,a_0 - 2\,a_1, 2\,a_0 + a_1)$. Let $S$ be the parallelogram with vertices $(1, -2), (3, 1), (4, 5), (6, 8)$.

(2 points) (a) Find the standard matrix of $T$.

(2 points) (b) Compute the area of $S$.

(2 points) (c) Compute the area of $T(S)$.

Sol: (a) $T = \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}$

(b) $Area = \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} = 5$

(c) $Area = \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} = 6 \cdot 5 = 30$

(6 points) 5. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be two bases of $\mathbb{R}^3$, where

$$[\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3] = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & -4 \\ -5 & -3 & -2 \end{bmatrix}, \quad [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3] = \begin{bmatrix} 1 & 1 & 1 \\ -3 & -4 & -3 \\ 0 & -1 & -1 \end{bmatrix}.$$

Compute the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Sol:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -3 & -4 & -3 & -2 & -3 & -4 \\ 0 & -1 & -1 & -5 & -3 & -2 \end{bmatrix} \sim_{\text{pivot at (1,1)}}^{r_2 \to r_2 + 3r_1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & -5 & -3 & -2 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_3 \to r_3 - r_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -6 & -3 & -1 \end{bmatrix}$$

$$\sim^{r_3 \to -r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 & 3 & 1 \end{bmatrix}$$

$$\sim_{\text{pivot at (3,3)}}^{r_1 \to r_1 - r_3} \begin{bmatrix} 1 & 1 & 0 & -5 & -2 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6 & 3 & 1 \end{bmatrix}$$

$$\sim^{r_2 \to -r_2} \begin{bmatrix} 1 & 1 & 0 & -5 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & 3 & 1 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_1 \to r_1 - r_2} \begin{bmatrix} 1 & 0 & 0 & -4 & -2 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & 3 & 1 \end{bmatrix}$$

The desired matrix is $\begin{bmatrix} -4 & -2 & -1 \\ -1 & 0 & 1 \\ 6 & 3 & 1 \end{bmatrix}$.

(6 points) 6. Diagonalize $A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$ if possible.

Sol: Step1: Compute the eigenvalues.

The characteristic polynomial is $\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 & -1 \\ 1 & -2-\lambda & -1 \\ -1 & 1 & -\lambda \end{bmatrix} = -\lambda\,(\lambda+1)^2$.

Thus the eigenvalues are $-1, 0$, with multiplicity $2, 1$, respectively.

Step 2: Compute the eigenvectors.

For $\lambda_1 = -1$, we have

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \sim^{\substack{r_2 \to r_2 - r_1 \\ r_3 \to r_3 + r_1}}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 0$, we have

$$\begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -1 \\ -1 & 1 & 0 \end{bmatrix} \sim^{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 + r_1}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 - r_2}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_2 \to -r_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to r_1 + 2\,r_2}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvectors are $v_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Step 3: construct $P$. Set $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

Step 4: construct $D$. Set $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Thus $A$ is diagonalizable and we have $A = PDP^{-1}$.

7. Suppose that $A$ is an $n \times n$ matrix. If $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ is a polynomial, then define $p(A) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3$. For instance if $p(t) = 1 - t^3$ then $p(A) = I - A^3$. Assume $p(A) = O$ (the zero matrix) for $p(t) = \left(t^2 - 2\right)(t+1)$.

(3 points) (a) Find all possible eigenvalues of $A$.

(3 points) (b) Show that $A$ is invertible and represent $A^{-1}$ in terms of $q(A)$ for a polynomial $q(t)$.

Sol: (a) $\sqrt{2}, -\sqrt{2}, -1$

(b) $p(A) = (A^2 - 2I)(A + I) = A^3 + A^2 - 2A - 2I = O$

Thus $I = \frac{A^3+A^2-2A}{2} = A \cdot \frac{A^2+A-2I}{2}$, and so $A^{-1} = \frac{A^2+A-2I}{2}$, i.e., $q(A) = \frac{A^2+A-2I}{2}$.

(6 points) 8. The columns of $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & -4 & 0 \\ 2 & -1 & -4 \\ 0 & 3 & -3 \end{bmatrix}$ are linearly independent. Find an orthonormal basis for Col$A$, the column space of $A$.

Sol: Step 1: Set $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{span}\{\mathbf{v}_1\}$.

Step 2: $\text{proj}_{W_1}\mathbf{x}_2 = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 0 \end{bmatrix}$,

Set $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$,

Step 3: $\text{proj}_{W_2}\mathbf{x}_3 = \begin{bmatrix} -3/2 \\ -1/2 \\ -2 \\ -3 \end{bmatrix}$,

Set $\mathbf{v}_3 = \begin{bmatrix} 5/2 \\ 1/2 \\ -2 \\ 0 \end{bmatrix}$,

Therefore, the desired orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 5/2 \\ 1/2 \\ -2 \\ 0 \end{bmatrix} \right\}$.

The desired orthonormal basis is $\left\{ \begin{bmatrix} 1/14\sqrt{14} \\ 3/14\sqrt{14} \\ 1/7\sqrt{14} \\ 0 \end{bmatrix} \begin{bmatrix} 1/6\sqrt{3} \\ -1/6\sqrt{3} \\ 1/6\sqrt{3} \\ 1/2\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{5}{42}\sqrt{42} \\ 1/42\sqrt{42} \\ -2/21\sqrt{42} \\ 0 \end{bmatrix} \right\}$.

9. Let $P$ be an $n \times n$ matrix satisfying $||P\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

(3 points) (a) Show that $P\mathbf{x} \cdot P\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ holds for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

(3 points) (b) Show that $P$ is an orthogonal matrix.

Sol: (a) Consider for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\begin{aligned}\|P(\mathbf{x}+\mathbf{y})\|^2 = \|P\mathbf{x}+P\mathbf{y}\|^2 &= (P\mathbf{x}+P\mathbf{y}) \cdot (P\mathbf{x}+P\mathbf{y}) \\ &= P\mathbf{x} \cdot P\mathbf{x} + P\mathbf{y} \cdot P\mathbf{y} + 2(P\mathbf{x} \cdot P\mathbf{y}) \\ &= \|P\mathbf{x}\|^2 + \|P\mathbf{y}\|^2 + 2(P\mathbf{x} \cdot P\mathbf{y}) = \|x\|^2 + \|y\|^2 + 2(P\mathbf{x} \cdot P\mathbf{y}) \qquad (1)\end{aligned}$$

Meanwhile, according to the condition,

$$\|P(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} + 2(\mathbf{x} \cdot \mathbf{y}) = \|x^2\| + \|y\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) \qquad (2)$$

Thus combining (1) and (2) we get $P\mathbf{x} \cdot P\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

(b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be the set of canonical bases of $\mathbb{R}^n$ such that $I_n = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$.
Also write $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$, where $\mathbf{p}_i$ denotes the i-th column of $P$.
Then for each $0 \le i, j \le n$, $\mathbf{p}_i = P \cdot \mathbf{e}_i$ and $\mathbf{p}_j = P \cdot \mathbf{e}_j$.
So by (a) we know

$$\mathbf{p}_i \cdot \mathbf{p}_j = (P \cdot \mathbf{e}_i) \cdot (P \cdot \mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus $P^T P = I_n$, which implies that $P$ is an orthonormal matrix. In particular, $P$ is an orthogonal matrix.

**Math 18D** **Final Exam ver. B** **Name:** ____________

**March 24, 2017** **(Total Points: *50*)** **PID:** ____________

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**Instructions**

1. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
2. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
3. **Read each question carefully and answer each question completely.**
4. **Show all of your work. No credit will be given for unsupported answers, even if correct.**
5. **Write your Name at the top of each page.**

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(1 point) 1. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 2. Determine if the following system of linear equations is consistent or not. If the system is consistent, describe the solution set by using parametric form.

$$\begin{cases} \quad\quad -x_2 \; -4x_3 \; +x_4 \; -2x_5 = -6 \\ x_1 \quad\quad +x_3 \; -2x_4 \; +7x_5 = 1 \\ -x_1 \; -2x_2 \; -9x_3 \; +5x_4 \; -13x_5 = -16 \end{cases}.$$

Sol:

$$\begin{bmatrix} 0 & -1 & -4 & 1 & -2 & -6 \\ 1 & 0 & 1 & -2 & 7 & 1 \\ -1 & -2 & -9 & 5 & -13 & -16 \end{bmatrix} \sim^{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 0 & 1 & -2 & 7 & 1 \\ 0 & -1 & -4 & 1 & -2 & -6 \\ -1 & -2 & -9 & 5 & -13 & -16 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 + r_1}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & 0 & 1 & -2 & 7 & 1 \\ 0 & -1 & -4 & 1 & -2 & -6 \\ 0 & -2 & -8 & 3 & -6 & -15 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 - 2r_2}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & 0 & 1 & -2 & 7 & 1 \\ 0 & -1 & -4 & 1 & -2 & -6 \\ 0 & 0 & 0 & 1 & -2 & -3 \end{bmatrix}$$

$$\sim^{\substack{r_1 \to r_1 + 2r_3 \\ r_2 \to r_2 - r_3}}_{\text{pivot at (3,4)}} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 & -5 \\ 0 & -1 & -4 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & -2 & -3 \end{bmatrix}$$

$$\sim^{r_2 \to -r_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 & -5 \\ 0 & 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 & -3 \end{bmatrix}$$

The general solution of the linear system is: $\begin{bmatrix} -5 \\ 3 \\ 0 \\ -3 \\ 0 \end{bmatrix} \begin{bmatrix} -6 \\ -1 \\ 1 \\ -3 \\ 0 \end{bmatrix} \begin{bmatrix} -8 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

The first vector is a special solution. See version A for the format.

3. Given that $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_6] = \begin{bmatrix} 1 & 1 & 0 & -3 & 0 & -2 \\ 5 & 4 & 0 & -15 & -1 & -12 \\ 1 & -2 & 1 & -8 & -3 & -11 \\ 1 & 2 & 2 & -13 & 2 & -1 \end{bmatrix}$ is row equivalent to $B = [\mathbf{b}_1\ \mathbf{b}_2 \cdots \mathbf{b}_6] = \begin{bmatrix} 1 & 0 & 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$. Let $H$ be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $K$ be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}$. Find a basis for each of the following subspaces.

(3 points) (a) $\text{Row}(A)^\perp$, the orthogonal complement of the row space of $A$.

(2 points) (b) $H + K$, the sum of the two subspaces $H$ and $K$.

(2 points) (c) $H \bigcap K$, the intersection of the two subspaces $H$ and $K$.

Sol: (a) The basis of $\text{Row}(A)^\perp$ is $\left\{ \begin{bmatrix} -1 \\ 3 \\ 3 \\ 0 \\ -5 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(b) The basis of $H + K$ is $\left\{ \begin{bmatrix} 1 \\ 5 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -2 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ -3 \\ 2 \end{bmatrix} \right\}$

(c) The basis of $H \bigcap K$ is $\{\mathbf{a}_4, \mathbf{a}_6 - 5\mathbf{a}_5\}$, or $\left\{ \begin{bmatrix} -3 \\ -15 \\ -8 \\ -13 \end{bmatrix} \begin{bmatrix} -2 \\ -7 \\ 4 \\ -11 \end{bmatrix} \right\}$

4. Let $T$ be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T(a_0, a_1) = (5\,a_0 - 2\,a_1, 5\,a_0 + a_1)$. Let $S$ be the parallelogram with vertices $(1, -2), (3, 1), (5, 5), (7, 8)$.

(2 points) (a) Find the standard matrix of $T$.

(2 points) (b) Compute the area of $S$.

(2 points) (c) Compute the area of $T(S)$.

Sol: (a) $T = \begin{bmatrix} 5 & -2 \\ 5 & 1 \end{bmatrix}$

(b) $Area = \begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} = 2$

(c) $Area = \begin{vmatrix} 5 & -2 \\ 5 & 1 \end{vmatrix} \cdot \begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} = 15 \cdot 2 = 30$

(6 points) 5. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be two bases of $\mathbb{R}^3$, where

$$[\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3] = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 5 \\ 2 & 5 & 8 \end{bmatrix}, \quad [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 1 & 3 \\ 1 & -2 & 6 \end{bmatrix}.$$

Compute the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Sol:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 5 & 1 & 3 & 3 & 2 & 5 \\ 1 & -2 & 6 & 2 & 5 & 8 \end{bmatrix} \underset{\text{pivot at (1,1)}}{\overset{\substack{r_2 \to r_2 - 5r_1 \\ r_3 \to r_3 - r_1}}{\sim}} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -2 & -3 & -5 \\ 0 & -2 & 5 & 1 & 4 & 6 \end{bmatrix}$$

$$\underset{\text{pivot at (2,2)}}{\overset{r_3 \to r_3 + 2r_2}{\sim}} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & -2 & -3 & -5 \\ 0 & 0 & 1 & -3 & -2 & -4 \end{bmatrix}$$

$$\underset{\text{pivot at (3,3)}}{\overset{\substack{r_1 \to r_1 - r_3 \\ r_2 \to r_2 + 2r_3}}{\sim}} \begin{bmatrix} 1 & 0 & 0 & 4 & 3 & 6 \\ 0 & 1 & 0 & -8 & -7 & -13 \\ 0 & 0 & 1 & -3 & -2 & -4 \end{bmatrix}$$

The desired matrix is $\begin{bmatrix} 4 & 3 & 6 \\ -8 & -7 & -13 \\ -3 & -2 & -4 \end{bmatrix}$.

(6 points) 6. Diagonalize $A = \begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 2 & -2 & 0 \end{bmatrix}$ if possible.

Sol: Step1: Compute the eigenvalues.

The characteristic polynomial is $\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & -2 & -1 \\ 2 & -1-\lambda & -1 \\ 2 & -2 & -\lambda \end{bmatrix} = -\lambda\,(\lambda-1)^2$.

Thus the eigenvalues are $0, 1$, with multiplicity $1, 2$, respectively.

Step 2: Compute the eigenvectors.

For $\lambda_1 = 0$, we have

$$\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 2 & -2 & 0 \end{bmatrix} \sim_{\text{pivot at (1,1)}}^{\substack{r_2 \to r_2 - 2/3\, r_1 \\ r_3 \to r_3 - 2/3\, r_1}} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1/3 & -1/3 \\ 0 & -2/3 & 2/3 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_3 \to r_3 + 2\, r_2} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1/3 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_2 \to 3\, r_2} \begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_1 \to r_1 + 2\, r_2} \begin{bmatrix} 3 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to 1/3\, r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 1$, we have

$$\begin{bmatrix} 2 & -2 & -1 \\ 2 & -2 & -1 \\ 2 & -2 & -1 \end{bmatrix} \sim_{\text{pivot at (1,1)}}^{\substack{r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_1}} \begin{bmatrix} 2 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to 1/2\, r_1} \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvectors are $v_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Step 3: construct $P$. Set $P = \begin{bmatrix} 1 & 1/2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Step 4: construct $D$. Set $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus $A$ is diagonalizable and we have $A = PDP^{-1}$.

7. Suppose that $A$ is an $n \times n$ matrix. If $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ is a polynomial, then define $p(A) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3$. For instance if $p(t) = 1 - t^3$ then $p(A) = I - A^3$. Assume $p(A) = O$ (the zero matrix) for $p(t) = \left(t^2 - 2\right)(t+3)$.

(3 points) (a) Find all possible eigenvalues of $A$.

(3 points) (b) Show that $A$ is invertible and represent $A^{-1}$ in terms of $q(A)$ for a polynomial $q(t)$.

(a) $\sqrt{2}, -\sqrt{2}, -3$

(b) $p(A) = (A^2 - 2I)(A + 3I) = A^3 + 3A^2 - 2A - 6I = O$

Thus $I = \frac{A^3+3A^2-2A}{6} = A \cdot \frac{A^2+3A-2I}{6}$, and so $A^{-1} = \frac{A^2+3A-2I}{6}$, i.e., $q(A) = \frac{A^2+3A-2I}{6}$.

(6 points) 8. The columns of $A = \begin{bmatrix} 2 & -3 & 2 \\ -1 & 3 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ are linearly independent. Find an orthonormal basis for Col$A$, the column space of $A$.

Sol: Step 1: Set $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{span}\{\mathbf{v}_1\}$.

Step 2: $\text{proj}_{W_1}\mathbf{x}_2 = \begin{bmatrix} -4 \\ 2 \\ 2 \\ 0 \end{bmatrix}$,

Set $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$,

Step 3: $\text{proj}_{W_2}\mathbf{x}_3 = \begin{bmatrix} 2 \\ -1/2 \\ -1/2 \\ 0 \end{bmatrix}$,

Set $\mathbf{v}_3 = \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \\ 4 \end{bmatrix}$,

Therefore, the desired orthogonal basis is $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \\ 4 \end{bmatrix} \right\}$.

The desired orthonormal basis is $\left\{ \begin{bmatrix} 1/3\sqrt{6} \\ -1/6\sqrt{6} \\ -1/6\sqrt{6} \\ 0 \end{bmatrix} \begin{bmatrix} 1/3\sqrt{3} \\ 1/3\sqrt{3} \\ 1/3\sqrt{3} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{1}{66}\sqrt{66} \\ \frac{1}{66}\sqrt{66} \\ \frac{4}{33}\sqrt{66} \end{bmatrix} \right\}$.

9. Let $Q$ be an $n \times n$ matrix satisfying $||Q\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

(3 points) (a) Show that $Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ holds for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

(3 points) (b) Show that $Q$ is an orthogonal matrix.

Sol: (a) Consider for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\begin{aligned}\|Q(\mathbf{x}+\mathbf{y})\|^2 = \|Q\mathbf{x}+Q\mathbf{y}\|^2 &= (Q\mathbf{x}+P\mathbf{y}) \cdot (Q\mathbf{x}+Q\mathbf{y}) \\ &= Q\mathbf{x} \cdot Q\mathbf{x} + Q\mathbf{y} \cdot Q\mathbf{y} + 2(Q\mathbf{x} \cdot Q\mathbf{y}) \\ &= \|Q\mathbf{x}\|^2 + \|Q\mathbf{y}\|^2 + 2(Q\mathbf{x} \cdot Q\mathbf{y}) = \|x\|^2 + \|y\|^2 + 2(Q\mathbf{x} \cdot Q\mathbf{y}) \qquad (1)\end{aligned}$$

Meanwhile, according to the condition,

$$\|Q(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x}+\mathbf{y}\cdot\mathbf{y}+2(\mathbf{x}\cdot\mathbf{y}) = \|x^2\|+\|y\|^2+2(\mathbf{x}\cdot\mathbf{y}) \qquad (2)$$

Thus combining (1) and (2) we get $Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

(b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be the set of canonical bases of $\mathbb{R}^n$ such that $I_n = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$.
Also write $Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix}$, where $\mathbf{q}_i$ denotes the i-th column of $Q$.
Then for each $0 \leq i, j \leq n$, $\mathbf{q}_i = Q \cdot \mathbf{e}_i$ and $\mathbf{q}_j = Q \cdot \mathbf{e}_j$.
So by (a) we know

$$\mathbf{q}_i \cdot \mathbf{q}_j = (Q \cdot \mathbf{e}_i) \cdot (Q \cdot \mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus $Q^T Q = I_n$, which implies that $Q$ is an orthonormal matrix. In particular, $Q$ is an orthogonal matrix.

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|---|---|---|
| **Math 18D** | **Final Exam ver. C** | **Name:** ____________ |
| **March 24, 2017** | **(Total Points: *50*)** | **PID:** ____________ |

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**Instructions**

1. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
2. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
3. **Read each question carefully and answer each question completely.**
4. **Show all of your work. No credit will be given for unsupported answers, even if correct.**
5. **Write your Name at the top of each page.**

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(1 point) 1. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 2. Determine if the following system of linear equations is consistent or not. If the system is consistent, describe the solution set by using parametric form.

$$\begin{cases} x_1 \quad\quad -2x_3 + 2x_4 = 5 \\ x_1 + x_2 + 2x_3 + x_4 - 2x_5 = -3 \\ x_1 + x_2 + 2x_3 + 2x_4 - 3x_5 = 0 \end{cases}.$$

Sol:

$$\begin{bmatrix} 1 & 0 & -2 & 2 & 0 & 5 \\ 1 & 1 & 2 & 1 & -2 & -3 \\ 1 & 1 & 2 & 2 & -3 & 0 \end{bmatrix} \underset{\text{pivot at (1,1)}}{\overset{\substack{r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_1}}{\sim}} \begin{bmatrix} 1 & 0 & -2 & 2 & 0 & 5 \\ 0 & 1 & 4 & -1 & -2 & -8 \\ 0 & 1 & 4 & 0 & -3 & -5 \end{bmatrix}$$

$$\underset{\text{pivot at (2,2)}}{\overset{r_3 \to r_3 - r_2}{\sim}} \begin{bmatrix} 1 & 0 & -2 & 2 & 0 & 5 \\ 0 & 1 & 4 & -1 & -2 & -8 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\underset{\text{pivot at (3,4)}}{\overset{\substack{r_1 \to r_1 - 2r_3 \\ r_2 \to r_2 + r_3}}{\sim}} \begin{bmatrix} 1 & 0 & -2 & 0 & 2 & -1 \\ 0 & 1 & 4 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

The general solution of the linear system is: $\begin{bmatrix} -1 \\ -5 \\ 0 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ -9 \\ 1 \\ 3 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$

The first vector is a special solution. See version A for the format.

3. Given that $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_6] = \begin{bmatrix} 1 & 1 & -2 & 0 & 2 & 0 \\ -2 & -3 & 3 & -2 & -5 & -2 \\ -1 & -3 & -1 & -4 & -3 & -5 \\ -3 & 0 & 11 & 6 & -6 & 7 \end{bmatrix}$ is row equivalent to $B = [\mathbf{b}_1\ \mathbf{b}_2 \cdots \mathbf{b}_6] = \begin{bmatrix} 1 & 0 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$. Let $H$ be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $K$ be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}$. Find a basis for each of the following subspaces.

(3 points) (a) $\mathrm{Row}(A)^\perp$, the orthogonal complement of the row space of $A$.

(2 points) (b) $H + K$, the sum of the two subspaces $H$ and $K$.

(2 points) (c) $H \bigcap K$, the intersection of the two subspaces $H$ and $K$.

Sol: (a) The basis of $\mathrm{Row}(A)^\perp$ is $\left\{ \begin{bmatrix} -3 \\ 1 \\ -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

(b) The basis of $H + K$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \\ -3 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ -3 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \\ 11 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ -3 \\ 6 \end{bmatrix} \right\}$

(c) The basis of $H \bigcap K$ is $\{\mathbf{a}_4, \mathbf{a}_6 - \mathbf{a}_5\}$, or $\left\{ \begin{bmatrix} 0 \\ -2 \\ -4 \\ 6 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -2 \\ 13 \end{bmatrix} \right\}$

4. Let $T$ be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T(a_0, a_1) = (a_0 - 9\,a_1, a_0 + 3\,a_1)$. Let $S$ be the parallelogram with vertices $(1, -3), (5, 0), (6, 4), (10, 7)$.

(2 points) (a) Find the standard matrix of $T$.

(2 points) (b) Compute the area of $S$.

(2 points) (c) Compute the area of $T(S)$.

Sol: (a) $T = \begin{bmatrix} 1 & -9 \\ 1 & 3 \end{bmatrix}$

(b) $Area = \begin{vmatrix} 4 & 5 \\ 3 & 7 \end{vmatrix} = 13$

(c) $Area = \begin{vmatrix} 1 & -9 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 4 & 5 \\ 3 & 7 \end{vmatrix} = 12 \cdot 13 = 156$

(6 points) 5. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be two bases of $\mathbb{R}^3$, where

$$[\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3] = \begin{bmatrix} 1 & -1 & 2 \\ -4 & 3 & -6 \\ 0 & -1 & 3 \end{bmatrix}, \quad [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3] = \begin{bmatrix} 1 & 1 & 1 \\ -3 & -3 & -4 \\ 5 & 6 & 7 \end{bmatrix}.$$

Compute the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Sol:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 2 \\ -3 & -3 & -4 & -4 & 3 & -6 \\ 5 & 6 & 7 & 0 & -1 & 3 \end{bmatrix} \sim^{\substack{r_2 \to r_2 + 3r_1 \\ r_3 \to r_3 - 5r_1}}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 2 & -5 & 4 & -7 \end{bmatrix}$$

$$\sim^{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -5 & 4 & -7 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_3 \to -r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -5 & 4 & -7 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\sim^{\substack{r_1 \to r_1 - r_3 \\ r_2 \to r_2 - 2r_3}}_{\text{pivot at (3,3)}} \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to r_1 - r_2}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & 0 & 0 & 7 & -5 & 9 \\ 0 & 1 & 0 & -7 & 4 & -7 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The desired matrix is $\begin{bmatrix} 7 & -5 & 9 \\ -7 & 4 & -7 \\ 1 & 0 & 0 \end{bmatrix}$.

(6 points) 6. Diagonalize $A = \begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 2 \\ -1 & -2 & -1 \end{bmatrix}$ if possible.

Sol: Step1: Compute the eigenvalues.

The characteristic polynomial is $\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -2 & -2 \\ 1 & 3-\lambda & 2 \\ -1 & -2 & -1-\lambda \end{bmatrix} = -\lambda\,(\lambda-1)^2$.

Thus the eigenvalues are $0, 1$, with multiplicity $1, 2$, respectively.

Step 2: Compute the eigenvectors.

For $\lambda_1 = 0$, we have

$$\begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 2 \\ -1 & -2 & -1 \end{bmatrix} \sim^{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -2 \\ -1 & -2 & -1 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 + r_1}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 + 1/2\, r_2}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_2 \to -1/2\, r_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to r_1 - 3\, r_2}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 1$, we have

$$\begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 2 \\ -1 & -2 & -2 \end{bmatrix} \sim^{\substack{r_2 \to r_2 + r_1 \\ r_3 \to r_3 - r_1}}_{\text{pivot at (1,1)}} \begin{bmatrix} -1 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to -r_1} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvectors are $v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

Step 3: construct $P$. Set $P = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Step 4: construct $D$. Set $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus $A$ is diagonalizable and we have $A = PDP^{-1}$.

7. Suppose that $A$ is an $n \times n$ matrix. If $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ is a polynomial, then define $p(A) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3$. For instance if $p(t) = 1 - t^3$ then $p(A) = I - A^3$. Assume $p(A) = O$ (the zero matrix) for $p(t) = \left(t^2 - 3\right)(t+1)$.

(3 points) (a) Find all possible eigenvalues of $A$.

(3 points) (b) Show that $A$ is invertible and represent $A^{-1}$ in terms of $q(A)$ for a polynomial $q(t)$.

(a) $\sqrt{3}, -\sqrt{3}, -1$

(b) $p(A) = (A^2 - 3I)(A + I) = A^3 + A^2 - 3A - 3I = O$

Thus $I = \frac{A^3+A^2-3A}{3} = A \cdot \frac{A^2+A-3I}{3}$, and so $A^{-1} = \frac{A^2+A-3I}{3}$, i.e., $q(A) = \frac{A^2+A-3I}{3}$.

(6 points) 8. The columns of $A = \begin{bmatrix} 1 & -6 & 4 \\ -2 & 9 & -3 \\ 0 & -3 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ are linearly independent. Find an orthonormal basis for Col$A$, the column space of $A$.

Sol: Step 1: Set $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{span}\{\mathbf{v}_1\}$.

Step 2: $\text{proj}_{W_1}\mathbf{x}_2 = \begin{bmatrix} -4 \\ 8 \\ 0 \\ -4 \end{bmatrix}$,

Set $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ -3 \\ 4 \end{bmatrix}$,

Step 3: $\text{proj}_{W_2}\mathbf{x}_3 = \begin{bmatrix} \frac{13}{5} \\ -\frac{19}{5} \\ 7/5 \\ -1/5 \end{bmatrix}$,

Set $\mathbf{v}_3 = \begin{bmatrix} 7/5 \\ 4/5 \\ -2/5 \\ 1/5 \end{bmatrix}$,

Therefore, the desired orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} 7/5 \\ 4/5 \\ -2/5 \\ 1/5 \end{bmatrix} \right\}$.

The desired orthonormal basis is $\left\{ \begin{bmatrix} 1/6\sqrt{6} \\ -1/3\sqrt{6} \\ 0 \\ 1/6\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/15\sqrt{30} \\ 1/30\sqrt{30} \\ -1/10\sqrt{30} \\ 2/15\sqrt{30} \end{bmatrix} \begin{bmatrix} 1/10\sqrt{70} \\ \frac{2}{35}\sqrt{70} \\ -1/35\sqrt{70} \\ \frac{1}{70}\sqrt{70} \end{bmatrix} \right\}$.

9. Let $R$ be an $n \times n$ matrix satisfying $||R\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

(3 points) (a) Show that $R\mathbf{x} \cdot R\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ holds for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

(3 points) (b) Show that $R$ is an orthogonal matrix.

Sol: (a) Consider for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\begin{aligned}\|R(\mathbf{x}+\mathbf{y})\|^2 = \|R\mathbf{x}+R\mathbf{y}\|^2 &= (R\mathbf{x}+R\mathbf{y})\cdot(R\mathbf{x}+R\mathbf{y}) \\ &= R\mathbf{x}\cdot R\mathbf{x} + R\mathbf{y}\cdot R\mathbf{y} + 2(R\mathbf{x}\cdot R\mathbf{y}) \\ &= \|R\mathbf{x}\|^2 + \|R\mathbf{y}\|^2 + 2(R\mathbf{x}\cdot R\mathbf{y}) = \|x\|^2 + \|y\|^2 + 2(R\mathbf{x}\cdot R\mathbf{y}) \quad (1)\end{aligned}$$

Meanwhile, according to the condition,

$$\|R(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x}+\mathbf{y}\cdot\mathbf{y}+2(\mathbf{x}\cdot\mathbf{y}) = \|x^2\|+\|y\|^2+2(\mathbf{x}\cdot\mathbf{y}) \quad (2)$$

Thus combining (1) and (2) we get $R\mathbf{x} \cdot R\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

(b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be the set of canonical bases of $\mathbb{R}^n$ such that $I_n = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$.
Also write $R = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_n \end{bmatrix}$, where $\mathbf{r}_i$ denotes the i-th column of $R$.
Then for each $0 \le i, j \le n$, $\mathbf{r}_i = R \cdot \mathbf{e}_i$ and $\mathbf{r}_j = R \cdot \mathbf{e}_j$.
So by (a) we know

$$\mathbf{r}_i \cdot \mathbf{r}_j = (R\cdot\mathbf{e}_i)\cdot(R\cdot\mathbf{e}_j) = \mathbf{e}_i\cdot\mathbf{e}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus $R^T R = I_n$, which implies that $R$ is an orthonormal matrix. In particular, $R$ is an orthogonal matrix.

**Math 18D** **Final Exam ver. D** **Name:** ______________

**March 24, 2017** **(Total Points: *50*)** **PID:** ______________

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**Instructions**

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5. **Write your Name at the top of each page.**

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(1 point) 1. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 2. Determine if the following system of linear equations is consistent or not. If the system is consistent, describe the solution set by using parametric form.

$$\begin{cases} x_1 + x_2 - 5x_3 - x_4 = -2 \\ x_1 + 2x_2 - 6x_3 - x_4 = -7 \\ x_1 + x_2 - 5x_3 - 3x_5 = -1 \end{cases}.$$

Sol:

$$\begin{bmatrix} 1 & 1 & -5 & -1 & 0 & -2 \\ 1 & 2 & -6 & -1 & 0 & -7 \\ 1 & 1 & -5 & 0 & -3 & -1 \end{bmatrix} \overset{\substack{r_2 \to r_2 - r_1 \\ r_3 \to r_3 - r_1}}{\sim_{\text{pivot at (1,1)}}} \begin{bmatrix} 1 & 1 & -5 & -1 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & -3 & 1 \end{bmatrix}$$

$$\overset{r_1 \to r_1 + r_3}{\sim_{\text{pivot at (3,4)}}} \begin{bmatrix} 1 & 1 & -5 & 0 & -3 & -1 \\ 0 & 1 & -1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & -3 & 1 \end{bmatrix}$$

$$\overset{r_1 \to r_1 - r_2}{\sim_{\text{pivot at (2,2)}}} \begin{bmatrix} 1 & 0 & -4 & 0 & -3 & 4 \\ 0 & 1 & -1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & -3 & 1 \end{bmatrix}$$

The general solution of the linear system is: $\begin{bmatrix} 4 \\ -5 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 8 \\ -4 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$

The first vector is a special solution. See version A for the format.

3. Given that $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_6] = \begin{bmatrix} 1 & 0 & -1 & -2 & 2 & 1 \\ -4 & 1 & 3 & 8 & -7 & -2 \\ -3 & -3 & 5 & 5 & -10 & -12 \\ -1 & -1 & 5 & 5 & 1 & 5 \end{bmatrix}$ is row equivalent to $B = [\mathbf{b}_1\ \mathbf{b}_2 \cdots \mathbf{b}_6] = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 7 \\ 0 & 1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$. Let $H$ be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $K$ be the subspace of $\mathbb{R}^4$ spanned by $\{\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}$. Find a basis for each of the following subspaces.

(3 points) (a) $\text{Row}(A)^\perp$, the orthogonal complement of the row space of $A$.

(2 points) (b) $H + K$, the sum of the two subspaces $H$ and $K$.

(2 points) (c) $H \bigcap K$, the intersection of the two subspaces $H$ and $K$.

Sol: (a)The basis of $\text{Row}(A)^\perp$ is $\{ \begin{bmatrix} -7 \\ -7 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \}$

(b) The basis of $H + K$ is $\{ \begin{bmatrix} 1 \\ -4 \\ -3 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -3 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 5 \\ 5 \end{bmatrix} \begin{bmatrix} 2 \\ -7 \\ -10 \\ 1 \end{bmatrix} \}$

(c) The basis of $H \bigcap K$ is $\{\mathbf{a}_4, \mathbf{a}_5 + \mathbf{a}_6\}$, or $\{ \begin{bmatrix} -2 \\ 8 \\ 5 \\ 5 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ -22 \\ 6 \end{bmatrix} \}$

4. Let $T$ be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T(a_0, a_1) = (5\,a_0 + 6\,a_1, 5\,a_0 - 3\,a_1)$. Let $S$ be the parallelogram with vertices $(-2,-1), (-1,2), (3,6), (4,9)$.

(2 points) (a) Find the standard matrix of $T$.

(2 points) (b) Compute the area of $S$.

(2 points) (c) Compute the area of $T(S)$.

Sol: (a) $T = \begin{bmatrix} 5 & 6 \\ 5 & -3 \end{bmatrix}$

(b) $\begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = \text{-8}$

Thus $Area = 8$.

(c) $\begin{vmatrix} 5 & 6 \\ 5 & -3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = (-45) \cdot (-8) = 360$

Thus $Area = 360$.

(6 points) 5. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be two bases of $\mathbb{R}^3$, where

$$[\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 5 & -1 & -5 \end{bmatrix}, \quad [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3] = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & 1 \\ 5 & -14 & -8 \end{bmatrix}.$$

Compute the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Sol:

$$\begin{bmatrix} 1 & -2 & -1 & 1 & -1 & 0 \\ -2 & 3 & 1 & -1 & 2 & -1 \\ 5 & -14 & -8 & 5 & -1 & -5 \end{bmatrix} \overset{\substack{r_2 \to r_2 + 2r_1 \\ r_3 \to r_3 - 5r_1}}{\sim}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & -2 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & -4 & -3 & 0 & 4 & -5 \end{bmatrix}$$

$$\overset{r_3 \to r_3 - 4r_2}{\sim}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & -2 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 & 4 & -1 \end{bmatrix}$$

$$\overset{\substack{r_1 \to r_1 + r_3 \\ r_2 \to r_2 + r_3}}{\sim}_{\text{pivot at (3,3)}} \begin{bmatrix} 1 & -2 & 0 & -3 & 3 & -1 \\ 0 & -1 & 0 & -3 & 4 & -2 \\ 0 & 0 & 1 & -4 & 4 & -1 \end{bmatrix}$$

$$\sim^{r_2 \to -r_2} \begin{bmatrix} 1 & -2 & 0 & -3 & 3 & -1 \\ 0 & 1 & 0 & 3 & -4 & 2 \\ 0 & 0 & 1 & -4 & 4 & -1 \end{bmatrix}$$

$$\overset{r_1 \to r_1 + 2r_2}{\sim}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & 0 & 0 & 3 & -5 & 3 \\ 0 & 1 & 0 & 3 & -4 & 2 \\ 0 & 0 & 1 & -4 & 4 & -1 \end{bmatrix}$$

The desired matrix is $\begin{bmatrix} 3 & -5 & 3 \\ 3 & -4 & 2 \\ -4 & 4 & -1 \end{bmatrix}$.

(6 points) 6. Diagonalize $A = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & 1 \\ -2 & -2 & 0 \end{bmatrix}$ if possible.

Sol: Step1: Compute the eigenvalues.

The characteristic polynomial is $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & -1 \\ -2 & -3-\lambda & 1 \\ -2 & -2 & -\lambda \end{bmatrix} = -\lambda\,(\lambda+1)^2$.

Thus the eigenvalues are $-1, 0$, with multiplicity $2, 1$, respectively.

Step 2: Compute the eigenvectors.

For $\lambda_1 = -1$, we have

$$\begin{bmatrix} 2 & 2 & -1 \\ -2 & -2 & 1 \\ -2 & -2 & 1 \end{bmatrix} \sim_{\text{pivot at (1,1)}}^{\substack{r_2 \to r_2 + r_1 \\ r_3 \to r_3 + r_1}} \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to 1/2\, r_1} \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda_2 = 0$, we have

$$\begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & 1 \\ -2 & -2 & 0 \end{bmatrix} \sim_{\text{pivot at (1,1)}}^{\substack{r_2 \to r_2 + 2\, r_1 \\ r_3 \to r_3 + 2\, r_1}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_3 \to r_3 - 2\, r_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_1 \to r_1 - 2\, r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The corresponding eigenvectors are $v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

Step 3: construct $P$. Set $P = \begin{bmatrix} 1/2 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

Step 4: construct $D$. Set $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Thus $A$ is diagonalizable and we have $A = PDP^{-1}$.

7. Suppose that $A$ is an $n \times n$ matrix. If $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ is a polynomial, then define $p(A) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3$. For instance if $p(t) = 1 - t^3$ then $p(A) = I - A^3$. Assume $p(A) = O$ (the zero matrix) for $p(t) = \left(t^2 - 3\right)(t+2)$.

(3 points) (a) Find all possible eigenvalues of $A$.

(3 points) (b) Show that $A$ is invertible and represent $A^{-1}$ in terms of $q(A)$ for a polynomial $q(t)$.

(a) $\sqrt{3}, -\sqrt{3}, -2$

(b) $p(A) = (A^2 - 3I)(A + 2I) = A^3 + 2A^2 - 3A - 6I = O$

Thus $I = \frac{A^3+2A^2-3A}{6} = A \cdot \frac{A^2+2A-3I}{6}$, and so $A^{-1} = \frac{A^2+2A-3I}{6}$, i.e., $q(A) = \frac{A^2+2A-3I}{6}$.

(6 points) 8. The columns of $A = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 0 \\ 1 & 2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$ are linearly independent. Find an orthonormal basis for Col$A$, the column space of $A$.

Sol: Step 1: Set $\mathbf{v}_1 = \mathbf{x}_1$ and $W_1 = \text{span}\{\mathbf{v}_1\}$.

Step 2: $\text{proj}_{W_1}\mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$,

Set $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 1 \\ -1 \end{bmatrix}$,

Step 3: $\text{proj}_{W_2}\mathbf{x}_3 = \begin{bmatrix} 8/3 \\ 4/3 \\ 4/3 \\ 0 \end{bmatrix}$,

Set $\mathbf{v}_3 = \begin{bmatrix} -2/3 \\ -4/3 \\ 8/3 \\ 0 \end{bmatrix}$,

Therefore, the desired orthogonal basis is $\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} -2/3 \\ -4/3 \\ 8/3 \\ 0 \end{bmatrix} \right\}$.

The desired orthonormal basis is $\left\{ \begin{bmatrix} 0 \\ 1/3\sqrt{6} \\ 1/6\sqrt{6} \\ 1/6\sqrt{6} \end{bmatrix} \begin{bmatrix} 2/3\sqrt{2} \\ 0 \\ 1/6\sqrt{2} \\ -1/6\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/21\sqrt{21} \\ -2/21\sqrt{21} \\ \frac{4}{21}\sqrt{21} \\ 0 \end{bmatrix} \right\}$.

9. Let $S$ be an $n \times n$ matrix satisfying $||S\mathbf{x}|| = ||\mathbf{x}||$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

(3 points) (a) Show that $S\mathbf{x} \cdot S\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ holds for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

(3 points) (b) Show that $S$ is an orthogonal matrix.

Sol: (a) Consider for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\begin{aligned}\|S(\mathbf{x}+\mathbf{y})\|^2 = \|S\mathbf{x}+S\mathbf{y}\|^2 &= (S\mathbf{x}+P\mathbf{y}) \cdot (S\mathbf{x}+S\mathbf{y}) \\ &= S\mathbf{x} \cdot S\mathbf{x} + S\mathbf{y} \cdot S\mathbf{y} + 2(S\mathbf{x} \cdot S\mathbf{y}) \\ &= \|S\mathbf{x}\|^2 + \|S\mathbf{y}\|^2 + 2(S\mathbf{x} \cdot S\mathbf{y}) = \|x\|^2 + \|y\|^2 + 2(S\mathbf{x} \cdot S\mathbf{y}) \quad (1)\end{aligned}$$

Meanwhile, according to the condition,

$$\|S(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} + 2(\mathbf{x} \cdot \mathbf{y}) = \|x^2\| + \|y\|^2 + 2(\mathbf{x} \cdot \mathbf{y}) \quad (2)$$

Thus combining (1) and (2) we get $S\mathbf{x} \cdot S\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

(b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be the set of canonical bases of $\mathbb{R}^n$ such that $I_n = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{bmatrix}$.
Also write $S = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_n \end{bmatrix}$, where $\mathbf{q}_i$ denotes the i-th column of $S$.
Then for each $0 \leq i, j \leq n$, $\mathbf{s}_i = S \cdot \mathbf{e}_i$ and $\mathbf{s}_j = S \cdot \mathbf{e}_j$.
So by (a) we know

$$\mathbf{s}_i \cdot \mathbf{s}_j = (S \cdot \mathbf{e}_i) \cdot (S \cdot \mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Thus $S^T S = I_n$, which implies that $S$ is an orthonormal matrix. In particular, $S$ is an orthogonal matrix.