Math 18D **Midterm Exam 1 ver. A** **Name:** ____________

February 1, 2017 **(Total Points: *25*)** **PID:** ____________

Instructions

1. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
2. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
3. **Read each question carefully and answer each question completely.**
4. **Show all of your work. No credit will be given for unsupported answers, even if correct.**
5. **Write your Name at the top of each page.**

(1 point) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 1. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & -2 \\ -5 & 7 & -3 \end{bmatrix}$

(a) Find the RREF (reduced row echelon form) of A.

Sol.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & -2 \\ -5 & 7 & -3 \end{bmatrix} \overset{\substack{r_2 \to r_2 + r_1 \\ r_3 \to r_3 + 5\,r_1}}{\sim_{\text{pivot at (1,1)}}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\overset{r_3 \to r_3 + 2\,r_2}{\sim_{\text{pivot at (2,2)}}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_2 \to -r_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{r_1 \to r_1 + r_2}{\sim_{\text{pivot at (2,2)}}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

The corresponding linear system is $\begin{cases} x_1 = & -2\,x_3 \\ x_2 = & -x_3 \\ x_3 = & x_3, \qquad \text{(free)} \end{cases}$

So the general solution is given by: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, where x_3 is arbitrary.

(6 points) 2. Determine if the following system of linear equations is consistent or not. If the system is consistent, describe the solution set by using parametric form.

$$\begin{cases} \quad\quad\quad\quad\quad\quad x_3 + 2x_4 = 1 \\ x_1 - 3x_2 + x_3 + 4x_4 = 1 \\ -x_1 + 3x_2 + 4x_3 + 6x_4 = 4 \end{cases}$$

Sol. We work on the corresponding augmented matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & -3 & 1 & 4 & 1 \\ -1 & 3 & 4 & 6 & 4 \end{bmatrix} \sim^{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -3 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ -1 & 3 & 4 & 6 & 4 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 + r_1}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & -3 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 10 & 5 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 - 5r_2}_{\text{pivot at (2,3)}} \begin{bmatrix} 1 & -3 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to r_1 - r_2}_{\text{pivot at (2,3)}} \begin{bmatrix} 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the RREF we write down the corresponding linear system: $\begin{cases} x_1 = & 3x_2 - 2x_4 \\ x_2 = & x_2 \\ x_3 = & -2x_4 + 1 \\ x_4 = & x_4 \end{cases}$. Thus

the general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$,

where x_2 and x_4 are free.

(6 points) 3. For each $k \in \mathbb{R}$, let S_k be the set of vectors in $\mathbb{R}^3$ given by $S_k = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ k \end{bmatrix} \right\}$.

For each of parts (a) - (c), find the value(s) of k for which S_k has the indicated property. Be sure to show how you arrived at each answer.

(a) S_k is linearly dependent.

Sol. We need to use the REF to answer the questions. We have

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -3 & 4 \\ 1 & 3 & k \end{bmatrix} \underset{\text{pivot at (1,1)}}{\overset{\substack{r_2 \to r_2 + 2r_1 \\ r_3 \to r_3 - r_1}}{\sim}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 1 & k+3 \end{bmatrix}$$

$$\underset{\text{pivot at (2,2)}}{\overset{r_3 \to r_3 - r_2}{\sim}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & k+5 \end{bmatrix}$$

Thus when $k = -5$, S_k is linearly dependent.

(b) S_k is linearly independent.

Sol. When $k \neq -5$, S_k is linearly independent.

(c) S_k spans $\mathbb{R}^3$.

Sol. When $k \neq -5$, S_k spans $\mathbb{R}^3$

(6 points) 4. Suppose that $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ are linear transformations. Moreover the standard matrices of T_1 and T_2 are given by

$$T_1(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & 1 \\ 4 & 3 & -2 \end{bmatrix} \mathbf{x}, \qquad T_2(\mathbf{x}) = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix} \mathbf{x}.$$

Find the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = T_2(T_1(\mathbf{x}))$.

Sol. If we write $T_1(\mathbf{x}) = A\mathbf{x}$ and $T_2(\mathbf{x}) = B\mathbf{x}$, then $T_2(T_1(\mathbf{x})) = T_2(A\mathbf{x}) = BA\mathbf{x}$. Thus BA is the standard matrix.

For our problem, we have

$$BA = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -5 & -4 & 1 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 0 \\ 39 & 30 & -15 \end{bmatrix}.$$

Alternatively, the standard matrix is given by $[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)]$. The computation is straightforward, so I omit the details.

Math 18D **Midterm Exam 1 ver. B** **Name:** ____________

February 1, 2017 **(Total Points: *25*)** **PID:** ____________

Instructions

1. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
2. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
3. **Read each question carefully and answer each question completely.**
4. **Show all of your work. No credit will be given for unsupported answers, even if correct.**
5. **Write your Name at the top of each page.**

(1 point) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 1. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & -3 \\ -5 & 7 & -1 \end{bmatrix}$

(a) Find the RREF (reduced row echelon form) of A.

Sol.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & -3 \\ -5 & 7 & -1 \end{bmatrix} \sim^{\substack{r_2 \to r_2 + r_1 \\ r_3 \to r_3 + 5\,r_1}}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 + 2\,r_2}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_2 \to -r_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to r_1 + r_2}_{\text{pivot at (2,2)}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

The corresponding linear system is $\begin{cases} x_1 = & -3\,x_3 \\ x_2 = & -2\,x_3 \\ x_3 = & x_3 \text{ (free)} \end{cases}$

So the general solution is given by: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$, where x_3 is arbitrary.

(6 points) 2. Determine if the following system of linear equations is consistent or not. If the system is consistent, describe the solution set by using parametric form.

$$\begin{cases} \quad\quad\quad\quad\quad\quad\quad x_3 + 2x_4 = 1 \\ x_1 - 3x_2 + x_3 + 4x_4 = 2 \\ -x_1 + 3x_2 + 4x_3 + 6x_4 = 3 \end{cases}$$

Sol. We work on the corresponding augmented matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & -3 & 1 & 4 & 2 \\ -1 & 3 & 4 & 6 & 3 \end{bmatrix} \sim^{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -3 & 1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ -1 & 3 & 4 & 6 & 3 \end{bmatrix}$$

$$\underset{\text{pivot at (1,1)}}{\overset{r_3 \to r_3 + r_1}{\sim}} \begin{bmatrix} 1 & -3 & 1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 10 & 5 \end{bmatrix}$$

$$\underset{\text{pivot at (2,3)}}{\overset{r_3 \to r_3 - 5r_2}{\sim}} \begin{bmatrix} 1 & -3 & 1 & 4 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\text{pivot at (2,3)}}{\overset{r_1 \to r_1 - r_2}{\sim}} \begin{bmatrix} 1 & -3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the RREF we write down the corresponding linear system: $\begin{cases} x_1 = & 3x_2 - 2x_4 + 1 \\ x_2 = & x_2 \\ x_3 = & -2x_4 + 1 \\ x_4 = & x_4 \end{cases}$.

Thus the general solution is: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

(6 points) 3. For each $k \in \mathbb{R}$, let S_k be the set of vectors in $\mathbb{R}^3$ given by $S_k = \left\{ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ k \end{bmatrix} \right\}$.

For each of parts (a) - (c), find the value(s) of k for which S_k has the indicated property. Be sure to show how you arrived at each answer.

(a) S_k is linearly dependent.

Sol. We need to use the REF to answer the questions. We have "

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -3 & 4 \\ 5 & 1 & k \end{bmatrix} \overset{\substack{r_2 \to r_2 + 2\,r_1 \\ r_3 \to r_3 - 5\,r_1}}{\sim_{\text{pivot at (1,1)}}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & -9 & k+15 \end{bmatrix}$$

$$\overset{r_3 \to r_3 + 9\,r_2}{\sim_{\text{pivot at (2,2)}}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & k-3 \end{bmatrix}$$

Thus when $k = 3$, S_k is linearly dependent.

(b) S_k is linearly independent.

Sol. When $k \neq 3$, S_k is linearly independent.

(c) S_k spans $\mathbb{R}^3$.

Sol. When $k \neq 3$, S_k spans $\mathbb{R}^3$

(6 points) 4. Suppose that $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ are linear transformations. Moreover the standard matrices of T_1 and T_2 are given by

$$T_1(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & 1 \\ 4 & 3 & -2 \end{bmatrix} \mathbf{x}, \qquad T_2(\mathbf{x}) = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & 2 \end{bmatrix} \mathbf{x}.$$

Find the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = T_2(T_1(\mathbf{x}))$.

Sol. If we write $T_1(\mathbf{x}) = A\mathbf{x}$ and $T_2(\mathbf{x}) = B\mathbf{x}$, then $T_2(T_1(\mathbf{x})) = T_2(A\mathbf{x}) = BA\mathbf{x}$. Thus BA is the standard matrix.

For our problem, we have

$$BA = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -5 & -4 & 1 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 1 \\ 31 & 24 & -13 \end{bmatrix}.$$

Alternatively, the standard matrix is given by $[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)]$. The computation is straightforward, so I omit the details.

Math 18D	**Midterm Exam 1 ver. C**	**Name:** ______________
February 1, 2017	**(Total Points: *25*)**	**PID:** ______________

Instructions

1. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
2. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
3. **Read each question carefully and answer each question completely.**
4. **Show all of your work. No credit will be given for unsupported answers, even if correct.**
5. **Write your Name at the top of each page.**

(1 point) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 1. Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 2 \\ -5 & 7 & 3 \end{bmatrix}$

(a) Find the RREF (reduced row echelon form) of A.

Sol.

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 2 \\ -5 & 7 & 3 \end{bmatrix} \sim_{\text{pivot at (1,1)}}^{\substack{r_2 \to r_2 + r_1 \\ r_3 \to r_3 + 5\,r_1}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_3 \to r_3 + 2\,r_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_2 \to -r_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_1 \to r_1 + r_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

The corresponding linear system is $\begin{cases} x_1 = & 2\,x_3 \\ x_2 = & x_3 \\ x_3 = & x_3 \text{ (free)} \end{cases}$

So the general solution is given by: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, where x_3 is arbitrary.

(6 points) 2. Determine if the following system of linear equations is consistent or not. If the system is consistent, describe the solution set by using parametric form.

$$\begin{cases} \quad\quad\quad\quad\quad\quad x_3 + 2x_4 = -1 \\ x_1 - 3x_2 + x_3 + 4x_4 = -1 \\ -x_1 + 3x_2 + 4x_3 + 6x_4 = -4 \end{cases}$$

Sol. We work on the corresponding augmented matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 2 & -1 \\ 1 & -3 & 1 & 4 & -1 \\ -1 & 3 & 4 & 6 & -4 \end{bmatrix} \sim^{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -3 & 1 & 4 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ -1 & 3 & 4 & 6 & -4 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 + r_1}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & -3 & 1 & 4 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 5 & 10 & -5 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 - 5r_2}_{\text{pivot at (2,3)}} \begin{bmatrix} 1 & -3 & 1 & 4 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to r_1 - r_2}_{\text{pivot at (2,3)}} \begin{bmatrix} 1 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the RREF we write down the corresponding linear system: $\begin{cases} x_1 = & 3x_2 - 2x_4 \\ x_2 = & x_2 \\ x_3 = & -2x_4 - 1 \\ x_4 = & x_4 \end{cases}$. Thus the general solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

(6 points) 3. For each $k \in \mathbb{R}$, let S_k be the set of vectors in $\mathbb{R}^3$ given by $S_k = \left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ k \end{bmatrix} \right\}$.

For each of parts (a) - (c), find the value(s) of k for which S_k has the indicated property. Be sure to show how you arrived at each answer.

(a) S_k is linearly dependent.

Sol. We need to use the REF to answer the questions. We have

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -3 & 4 \\ 4 & 1 & k \end{bmatrix} \overset{\substack{r_2 \to r_2 + 2\,r_1 \\ r_3 \to r_3 - 4\,r_1}}{\sim_{\text{pivot at (1,1)}}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & -7 & k+12 \end{bmatrix}$$

$$\overset{r_3 \to r_3 + 7\,r_2}{\sim_{\text{pivot at (2,2)}}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & k-2 \end{bmatrix}$$

Thus when $k = 2$, S_k is linearly dependent.

(b) S_k is linearly independent.

Sol. When $k \neq 2$, S_k is linearly independent.

(c) S_k spans $\mathbb{R}^3$.

Sol. When $k \neq 2$, S_k spans $\mathbb{R}^3$

(6 points) 4. Suppose that $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ are linear transformations. Moreover the standard matrices of T_1 and T_2 are given by

$$T_1(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 3 \\ -5 & -4 & 1 \\ 4 & 3 & -2 \end{bmatrix} \mathbf{x}, \qquad T_2(\mathbf{x}) = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -2 \end{bmatrix} \mathbf{x}.$$

Find the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = T_2(T_1(\mathbf{x}))$.

Sol. If we write $T_1(\mathbf{x}) = A\mathbf{x}$ and $T_2(\mathbf{x}) = B\mathbf{x}$, then $T_2(T_1(\mathbf{x})) = T_2(A\mathbf{x}) = BA\mathbf{x}$. Thus BA is the standard matrix.

For our problem, we have

$$BA = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ -5 & -4 & 1 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 2 \\ 15 & 12 & -7 \end{bmatrix}.$$

Alternatively, the standard matrix is given by $[T(\mathbf{e}_1)\; T(\mathbf{e}_2)\; T(\mathbf{e}_3)]$. The computation is straightforward, so I omit the details.

Math 18D **Midterm Exam 1 ver. D** **Name:** ____________________

February 1, 2017 **(Total Points: *25*)** **PID:** ____________________

Instructions

1. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
2. **You may use one handwritten page of notes, but no books or other assistance during this exam.**
3. **Read each question carefully and answer each question completely.**
4. **Show all of your work. No credit will be given for unsupported answers, even if correct.**
5. **Write your Name at the top of each page.**

(1 point) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 1. Let $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 3 \\ -5 & 7 & 1 \end{bmatrix}$

(a) Find the RREF (reduced row echelon form) of A.

Sol.

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 3 \\ -5 & 7 & 1 \end{bmatrix} \sim_{\text{pivot at (1,1)}}^{\substack{r_2 \to r_2 + r_1 \\ r_3 \to r_3 + 5\,r_1}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_3 \to r_3 + 2\,r_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_2 \to -r_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim_{\text{pivot at (2,2)}}^{r_1 \to r_1 + r_2} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Describe the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

The corresponding linear system is $\begin{cases} x_1 = & 3\,x_3 \\ x_2 = & 2x_3 \\ x_3 = & x_3 \text{ (free)} \end{cases}$

So the general solution is given by: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, where x_3 is arbitrary.

(6 points) 2. Determine if the following system of linear equations is consistent or not. If the system is consistent, describe the solution set by using parametric form.

$$\begin{cases} \quad\quad\quad\quad\quad\quad x_3 + 2\,x_4 = -1 \\ x_1 - 3\,x_2 + x_3 + 4\,x_4 = -2 \\ -x_1 + 3\,x_2 + 4\,x_3 + 6\,x_4 = -3 \end{cases}$$

Sol. We work on the corresponding augmented matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 2 & -1 \\ 1 & -3 & 1 & 4 & -2 \\ -1 & 3 & 4 & 6 & -3 \end{bmatrix} \sim^{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -3 & 1 & 4 & -2 \\ 0 & 0 & 1 & 2 & -1 \\ -1 & 3 & 4 & 6 & -3 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 + r_1}_{\text{pivot at (1,1)}} \begin{bmatrix} 1 & -3 & 1 & 4 & -2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 5 & 10 & -5 \end{bmatrix}$$

$$\sim^{r_3 \to r_3 - 5\,r_2}_{\text{pivot at (2,3)}} \begin{bmatrix} 1 & -3 & 1 & 4 & -2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim^{r_1 \to r_1 - r_2}_{\text{pivot at (2,3)}} \begin{bmatrix} 1 & -3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the RREF we write down the corresponding linear system: $\begin{cases} x_1 = & 3\,x_2 - 2\,x_4 - 1 \\ x_2 = & x_2 \\ x_3 = & -2\,x_4 - 1 \\ x_4 = & x_4 \end{cases}$.

Thus the general solution is: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

(6 points) 3. For each $k \in \mathbb{R}$, let S_k be the set of vectors in $\mathbb{R}^3$ given by $S_k = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ k \end{bmatrix} \right\}$.

For each of parts (a) - (c), find the value(s) of k for which S_k has the indicated property. Be sure to show how you arrived at each answer.

(a) S_k is linearly dependent.

Sol. We need to use the REF to answer the questions. We have

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -3 & 4 \\ 1 & 2 & k \end{bmatrix} \underset{\text{pivot at (1,1)}}{\overset{\substack{r_2 \to r_2 + 2\, r_1 \\ r_3 \to r_3 - r_1}}{\sim}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & k+3 \end{bmatrix}$$

Thus when $k = -3$, S_k is linearly dependent.

(b) S_k is linearly independent.

Sol. When $k \neq -3$, S_k is linearly independent.

(c) S_k spans $\mathbb{R}^3$.

Sol. When $k \neq -3$, S_k spans $\mathbb{R}^3$

(6 points) 4. Suppose that $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ are linear transformations. Moreover the standard matrices of T_1 and T_2 are given by

$$T_1(\mathbf{x}) = \begin{bmatrix} 1 & 1 & 4 \\ -5 & -4 & 1 \\ 4 & 3 & -2 \end{bmatrix} \mathbf{x}, \qquad T_2(\mathbf{x}) = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -4 \end{bmatrix} \mathbf{x}.$$

Find the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = T_2(T_1(\mathbf{x}))$.

Sol. If we write $T_1(\mathbf{x}) = A\mathbf{x}$ and $T_2(\mathbf{x}) = B\mathbf{x}$, then $T_2(T_1(\mathbf{x})) = T_2(A\mathbf{x}) = BA\mathbf{x}$. Thus BA is the standard matrix.

For our problem, we have

$$BA = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -5 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 \\ -5 & -4 & 1 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 3 \\ 7 & 6 & -5 \end{bmatrix}.$$

Alternatively, the standard matrix is given by $[T(\mathbf{e}_1)\ T(\mathbf{e}_2)\ T(\mathbf{e}_3)]$. The computation is straightforward, so I omit the details.