Math 18: Lectures ABC | **Final Exam v. 1** | **Name:** ______________
March 17, 2018 | **(Total Points: *50*)** | **PID:** ______________

Instructions

1. **Write your Name and PID in the spaces provided above.**
2. **Make sure your Name is on every page.**
3. **No calculators, tablets, phones, or other electronic devices are allowed during this exam.**
4. **You may use two handwritten pages of notes, but no books or other assistance during this exam.**
5. **Read each question carefully and answer each question completely.**
6. **Write your solutions clearly in the spaces provided.**
7. **Show all of your work. No credit will be given for unsupported answers, even if correct.**

(1 point) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 1. (a) Let $A = \begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 3 & 1 & 1 & -1 & h \\ 0 & 1 & -5 & 2 & -5 \end{bmatrix}$. Determine all values of h for which A is *not* full rank.

What is the dimension of the null space of A for those values of h?

(b) Find the solution (in parametric form) of the following system of equations:

$$\begin{aligned} x_1 \qquad\; + 2x_3 &= -1 \\ 3x_1 + x_2 + \;\; x_3 &= -1 \\ x_2 - 5x_3 &= \;\; 2 \end{aligned}$$

(6 points) 2. The matrix $A = \begin{bmatrix} 2 & 3 & 1 & 8 & 2 & 3 \\ -1 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & -2 & 5 & 3 & 8 \\ 1 & 4 & 3 & 9 & 2 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for $\mathrm{Col}(A)$.

(b) Find a basis for $\mathrm{Row}(A)$.

(c) Find a basis for $\mathrm{Row}(A)^{\perp}$.

(10 points) 3. In each of the following examples, a vector space V is given, along with a subset $S \subseteq V$. Indicate whether the set S is a subspace of V or not, by filling in the ◯ **YES** bubble if it is a subspace, or the ◯ **NO** bubble if is not. 2 points will be assigned for each correct response, 1 point for each blank non-response, and 0 points for each incorrect response. No justification is required.

(a) $V = \mathbb{R}^2$, and $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \leq y \right\}$.

◯ **YES** ◯ **NO**

(b) $V = \mathbb{R}^n$, $W \subseteq V$ is a subspace, and $S = W^\perp$.

◯ **YES** ◯ **NO**

(c) $V = \mathbb{R}^3$, and S is the set of vectors in V with length 1: $S = \{\mathbf{v} \in V : \|\mathbf{v}\| = 1\}$.

◯ **YES** ◯ **NO**

(d) $V = M_{4\times4}$ is the space of 4×4 matrices, and S is the set of *skew-symmetric* matrices: $S = \{A \in M_{4\times4} : A^\top = -A\}$.

◯ **YES** ◯ **NO**

(e) $V = \mathbb{P}_3$ is the space of polynomials of degree ≤ 3, and $S = \{\mathbf{p} \in \mathbb{P}_3 : \mathbf{p}(1) = 1\}$.

◯ **YES** ◯ **NO**

(8 points) 4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

(a) Find all the eigenvalues of A. Is A diagonalizable? Explain your answer.

(b) Determine the eigenspace of A for the *largest* eigenvalue, 3.

(c) What is the largest eigenvalue of the matrix A^2? And what is its eigenspace?

(7 points) 5. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

(a) Calculate the determinant of A, showing your work. Use this to show that A is invertible.

(b) Calculate the inverse of A, showing your work.

(c) Explain why there is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

(6 points) 6. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 3 \\ -2 & -1 \end{bmatrix}$.

(a) Use the Gram–Schmidt process to find an orthonormal basis $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2\}$ for $\mathrm{Col}(A)$.

(b) Use the result in part (a) to factorize $A = QR$ where $Q \in M_{4\times 2}$ has orthonormal columns and $R \in M_{2\times 2}$ is upper-triangular.

(6 points) 7. Suppose A and B are square matrices, and suppose that $AB = BA$. Let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ.

(a) Show that $\mathbf{u} = B\mathbf{v}$ is also an eigenvector of A, with eigenvalue λ.

(b) Let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ, as above. Suppose that the algebraic multiplicity of the eigenvalue λ is 1. Use (a) to show that $\mathbf{v}$ is also an eigenvector of B.