1. Consider the matrices $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$.

(a) Find $\det A$ and $\det B$.

(b) Find $\det A^{-1}$ and $\det A^2B$.

(c) Is A^3B^3 invertible?

2. Find a basis for Row A, Nul A and for Col A, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}.$$

3. (a) If a 4 x 5 matrix A has rank 2, find dimRow(A).

(b) If a 5 x 6 matrix A has rank 3, find Rank A^T.

(c) If A is a 7 x 9 matrix, what is the largest possible rank of A?

4. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases for the vector space $\mathbb{R}^2$.

(a) If $\mathcal{B} = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ and $\mathcal{C} = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\}$, find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Prove that the inverse of the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

5. Let A and B be two n x n matrices.

   (a) Prove that Col $(AB) \subset$ Col A.

   (b) If B is invertible, prove that Col $(AB) =$ Col A.

6. Let $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$. (*You should try this problem after Tuesday, after we cover eigenvalues.*)

(a) Find the eigenvalues of A and the associate eigenvectors.

(b) Prove that A is diagonalizable and find P invertible D diagonal such that $A = PDP^{-1}$.

7. (a) Let A be an n x n matrix such that $Ax = 0$ for all $x \in \mathbb{R}^n$. Prove that $A^T x = 0$ for all $x \in \mathbb{R}^n$.

(b) Let A be a $n \times n$ matrix with real entries such that $A^T A = 0$. Prove that $A = 0$.

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(c) Let A be a 2×2 matrix such that $A^2 = I_2$. Is it true that $A = I_2$? Justify your answer.

8. Let $\mathbb{M}_2(\mathbb{R})$ be the vector space of all 2x2 matrices with real entries. Denote by

$$H = \{A \in \mathbb{M}_2(\mathbb{R}); A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A\}$$

(a) Prove that H is a subspace of $\mathbb{M}_2(\mathbb{R})$.

(b) Find a basis of H and the dimension of H.

(c) Find $A, B \in \mathbb{M}_2(\mathbb{R})$ such that $A \notin H$, $B \notin H$ and $AB \neq BA$.

HINTS:

1) For part (b), use Theorem 6, section 3.2. For (c), use Theorems 4 and 6, section 3.2.
3) For part (b), use that dim Row $A=$ dim Col A^T. You actually obtain using The Rank Theorem that $Rank A = Rank A^T$. For (c) the largest possible rank is 7 (why?).
4) Use Theorem 15, section 4.7 for (b).
5) Use the fact that if $A = [a_1\, a_2 \ldots, a_n]$ then $Ax = x_1 a_1 + \ldots + x_n a_n$.
7) For (a), take x to be a vector e_i from the standard basis of $\mathbb{R}^n$. Conclude that actually $A = 0$. For (b), note that each of the diagonal entries of the matrix $A^T A$ is a sum of squares. If a sum of non-negative numbers is 0, then all those numbers are 0. For (c), try to find a counter example.
8) Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and note that H is the solution set of a linear system.