1. Consider the matrices \( A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix} \).

(a) Find \( \det A \) and \( \det B \).

(b) Find \( \det A^{-1} \) and \( \det A^2B \).

(c) Is \( A^3B^3 \) invertible?
2. Find a basis for Row $A$, Nul $A$ and for Col $A$, where

$$A = \begin{bmatrix}
1 & 0 & -1 \\
2 & 1 & -1 \\
1 & 1 & 2
\end{bmatrix}.$$
3. (a) If a $4 \times 5$ matrix $A$ has rank 2, find $\text{dimRow}(A)$. 

(b) If a $5 \times 6$ matrix $A$ has rank 3, find $\text{Rank } A^T$. 

(c) If $A$ is a $7 \times 9$ matrix, what is the largest possible rank of $A$?
4. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases for the vector space $\mathbb{R}^2$.

(a) If $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \}$ and $\mathcal{C} = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \}$, find the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

(b) Prove that the inverse of the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is the change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.
5. Let $A$ and $B$ be two $n \times n$ matrices.

(a) Prove that $\text{Col} \ (AB) \subset \text{Col} \ A$.

(b) If $B$ is invertible, prove that $\text{Col} \ (AB) = \text{Col} \ A$. 
6. Let $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$. (You should try this problem after Tuesday, after we cover eigenvalues.)

(a) Find the eigenvalues of $A$ and the associate eigenvectors.

(b) Prove that $A$ is diagonalizable and find $P$ invertible $D$ diagonal such that $A = PDP^{-1}$. 
7. (a) Let $A$ be an $n \times n$ matrix such that $Ax = 0$ for all $x \in \mathbb{R}^n$. Prove that $A^T x = 0$ for all $x \in \mathbb{R}^n$.

(b) Let $A$ be a $n \times n$ matrix with real entries such that $A^T A = 0$. Prove that $A = 0$.

(c) Let $A$ be a $2 \times 2$ matrix such that $A^2 = I_2$. Is it true that $A = I_2$? Justify your answer.
8. Let $M_2(\mathbb{R})$ be the vector space of all 2x2 matrices with real entries. Denote by

$$H = \{ A \in M_2(\mathbb{R}); A \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A \}$$

(a) Prove that $H$ is a subspace of $M_2(\mathbb{R})$.

(b) Find a basis of $H$ and the dimension of $H$.

(c) Find $A, B \in M_2(\mathbb{R})$ such that $A \notin H$, $B \notin H$ and $AB \neq BA$. 
HINTS:

1) For part (b), use Theorem 6, section 3.2. For (c), use Theorems 4 and 6, section 3.2.

3) For part (b), use that $\text{dim Row } A = \text{dim Col } A^T$. You actually obtain using The Rank Theorem that $\text{Rank } A = \text{Rank } A^T$. For (c) the largest possible rank is 7 (why?).

4) Use Theorem 15, section 4.7 for (b).

5) Use the fact that if $A = [a_1 \ a_2 \ldots \ a_n]$ then $Ax = x_1 a_1 + \ldots + x_n a_n$.

7) For (a), take $x$ to be a vector $e_i$ from the standard basis of $\mathbb{R}^n$. Conclude that actually $A = 0$. For (b), note that each of the diagonal entries of the matrix $A^T A$ is a sum of squares. If a sum of non-negative numbers is 0, then all those numbers are 0. For (c), try to find a counter example.

8) Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and note that $H$ is the solution set of a linear system.