Solutions to Math 21C Final, Winter 02.

1. Two vectors for two of the sides of the triangle are \( \mathbf{a} = (1, 2, 1) \) and \( \mathbf{b} = (2, -3, 2) \). The area of the triangle is \( |\mathbf{a} \times \mathbf{b}|/2 \). We have \( \mathbf{a} \times \mathbf{b} = \cdots = 8 \mathbf{i} - \mathbf{j} - 6 \mathbf{k} \) and \( |\mathbf{a} \times \mathbf{b}| = \sqrt{8^2 + 1 + 6^2} = \sqrt{101} \).

2. The curves intersect when \( (t, t^2, t^3) = (1 + s, 4s, 8s^2) \) for some \( t \) and \( s \). This means that \( t = 1 + s \) and \( t^2 = 4s = 4(t-1) \) so \( t^2 - 4t + 4 = 0 \) and hence \( (t-2)^2 = 0 \), i.e. \( t = 2 \) and hence \( s = 1 \). It is easy to check that also the last equation is satisfied for these values. We have \( \mathbf{r}_1'(t) = (1, 2t, 3t^2) \) so \( \mathbf{r}_1(2) = (1, 4, 12) \) and \( \mathbf{r}_2'(s) = (1, 4, 16s) \) so \( \mathbf{r}_2(1) = (1, 4, 16) \). The angle is given by \( \cos \theta = \mathbf{r}_1'(2) \cdot \mathbf{r}_2'(1) / (|\mathbf{r}_1'(2)| |\mathbf{r}_2'(1)|) = 209/(\sqrt{1 + 4^2 + 12^2} \cdot \sqrt{1 + 4^2 + 16^2}) = 209/161 \cdot 273 \).

3. The vector \( \mathbf{n} = (2, 1, 4) \) between the points is normal to the plane and the point in the middle between the points \( P = (0, 3/2, 1) \) on the plane. The equation of the plane is therefore \( 2(x-0) + 1(y-3/2) + 4(z-1) = 0 \).

4. \( \mathbf{r}'(t) = 3t^{1/2} \mathbf{i} - 2 \sin 2t \mathbf{j} + 2 \cos 2t \mathbf{k} \). The initial speed is \( |\mathbf{r}'(0)| = |2\mathbf{k}| = 2 \). The arc length is \( \int_0^1 |\mathbf{r}'(t)| \, dt = \int_0^1 \sqrt{9t + 4} \, dt = (9t + 4)^{3/2}/27|_0^1 = (13^3/2 - 4^3)/27 \).

5. The tangent plane to \( F(x, y, z) = x^2 + y^2/4 + z^2/9 = 1 \) at a point \( (x_0, y_0, z_0) \) has normal \( \nabla F(x_0, y_0, z_0) = (2x_0, y_0/2, 2z_0/9) \). This vector is parallel to the normal of the plane \( x+y-z=0 \), which is \( (1, 1, -1) \), if \( 2x_0 = \lambda, y_0/2 = \lambda \) and \( 2z_0/9 = -\lambda \) since the point also must lie on the surface we must have \( F(\lambda/2, \lambda, -9\lambda/2) = \lambda^2/4 + \lambda^2 + 9\lambda^2/4 = 7\lambda^2/2 = 1 \) so \( \lambda = \pm \sqrt{2}/7 \). The point is \( (x_0, y_0, z_0) = \pm \sqrt{2}/7 (1, 2, -9/2) \).

6. \( f_x(x, y) = 4x^2 - 8y = 0 \) and \( f_y(x, y) = -8x + 4y = 0 \) gives \( y = 2x \) and \( 4x(x^2-4) = 0 \). Hence \( x = 0 \) or \( x = \pm 2 \) so the critical points are \( (0, 0), (2, 4), (-2, -4) \). \( f_{xx} = 12x^2 \), \( f_{xy} = -8 \), \( f_{yy} = 4 \) so \( D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 48x^2 - 64 \). Then \( D(0, 0) < 0 \) so \( (0, 0) \) is a saddle point, \( D(2, 4) = 128 > 0 \) and \( f_{xx}(2, 4) = 48 > 0 \) so \( (2, 4) \) is local min, \( D(-2, -4) = 128 > 0 \) and \( f_{xx}(-2, -4) = 48 > 0 \) so \( (-2, -4) \) is local min.

7. Minimize the area \( A = xy+2xz+2yz \), subject to the constraint that the volume is \( V = xyz = 32,000 \). Lagrange multiplies: \( \nabla A(x, y, z) = \langle y + 2z, x + 2z, 2x + 2y \rangle \) and \( \nabla V(x, y, z) = \langle yz, xz, xy \rangle \) so we must find all \( (x, y, z) \) and \( \lambda \) such that \( y + 2z = \lambda yz, x + 2z = \lambda xz, 2x + 2y = \lambda xy \) and \( V(x, y, z) = 32,000 \). Multiplying the first equation by \( x \) and the second by \( y \) and the third by \( z \) we get \( x(y + 2z) = y(x + 2z) = z(2x + 2y) \). Subtracting the first two equations gives \( 2z(x - y) = 0 \) and subtracting the first and third gives \( (x - 2z)y = 0 \). If \( z = 0 \) then \( y = 0 \) or \( x = 0 \). If \( z \neq 0 \) then \( x = y = 0 \) or \( x = y = 2z \). Hence we have the points \( (x, 0, 0), (0, y, 0), (0, 0, z) \) and \( (2z, 2z, z) \). Only the last one gives \( V \neq 0 \) and we must have \( V(2z, 2z, z) = 4z^3 = 32,000 \) which is equivalent to \( z = 20 \) so \( (x, y, z) = (40, 40, 20) \).

8. Let \( D = \{(x, y); 10 - 3x^2 - 3y^2 \geq 4 \} = \{(x, y); x^2 + y^2 \leq 2 \} \).

The volume is \( \int_D z \, dA = \int_D (10 - 3x^2 - 3y^2) \, dA = \int_0^{2\pi} \int_0^{\sqrt{10}} (10 - 3r^2) \, r \, dr \, d\theta = \int_0^{2\pi} \frac{5r^2 - 3r^4}{4} \bigg|_0^{\sqrt{10}} d\theta = \int_0^{2\pi} \frac{7}{2} \, d\theta = 14\pi \).

9. \( \int_T \sqrt{1 + 1 + 4y^2} \, dA = \int_0^1 \int_0^y \sqrt{2 + 4y^2} \, dx \, dy = \int_0^1 \sqrt{2 + 4y^2} \, dy = \int_0^1 \sqrt{4y^2} \, dy = \int_0^{3/2} \sqrt{2} \, dy = (6^{3/2} - 2^{3/2})/12 \).

10. \( E = \{(x, y, z); x \geq 0, y \geq 0, z \geq 0, x + y + z/2 \leq 1 \} = \{(x, y, z); 0 \leq z \leq 2, 0 \leq y \leq 1 - z/2, 0 \leq x \leq 1 - y - z/2 \} \).

\( \int_E y \, dV = \int_0^1 \int_0^{1-z/2} \int_0^{1-y-z/2} y \, dx \, dy \, dz = \int_0^1 \int_0^{1-z/2} y \int_0^{1-y-z/2} dx \, dy \, dz = \int_0^1 \frac{1}{2} \int_0^{1-y-z/2} dy \int_0^{1-y-z/2} dz = \int_0^1 \frac{1}{2} \int_0^{1-z/2} y(1 - y - z/2) \, dy \, dz = \int_0^1 (2y^2/2 - y^3/3 - zy^2/4) \bigg|_{y=0}^{1-z/2} dz = \int_0^1 (1 - z/2)^3/6 \, dz = \int_0^1 t^{3/2} \, dt = t^{4/2}|_0^{1/2} = 1/12 \).