Solutions for Math 21C Midterm II, Fall 02, Lindblad.

1. (a) Velocity \( \mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + 3t^{1/2}/2 \mathbf{k} \) and speed

\[
|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + (3/2)^2 t} = \sqrt{1 + (3/2)^2 t}.
\]

Hence \( \mathbf{r}'(\pi) = -\mathbf{j} + (3/2)\sqrt{\pi} \mathbf{k} \) and \( |\mathbf{r}'(\pi)| = \sqrt{1 + (3/2)^2 \pi} \).

(b) The distance traveled is the arc length:

\[
\int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 \sqrt{1 + (3/2)^2 t} dt = (1 + (3/2)^2 t)^{3/2}(\frac{2}{3})^{3/2} |_0^1 = \left(1 + (\frac{3}{2})^2 10\right)^{3/2} - 1 \left(\frac{2}{3}\right)^3.
\]

2. (a) \( \nabla F(x, y) = (3x^2 - 2x, 2y - 1) \).

(b) Unit vector in the direction: \( \mathbf{u} = \langle 1, 2 \rangle / |\langle 1, 2 \rangle| = \langle 1, 2 \rangle / \sqrt{5} \). \( \nabla F(2, 3) = \langle 8, 5 \rangle \). The directional derivative \( D_\mathbf{u} F(2, 3) = \nabla F(2, 3) \cdot \mathbf{u} = \langle 8, 5 \rangle \cdot \langle 1, 2 \rangle / \sqrt{5} = 18 / \sqrt{5} \).

(c) By the chain rule \( h'(t) = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \) so \( h'(0) = \nabla F(1, 0) \cdot \langle 1, 1 \rangle = \langle 1, -1 \rangle \cdot \langle 1, 1 \rangle = 0 \)

3. (a) \( \langle f_x, f_y \rangle = \langle x, 3y \rangle / \sqrt{x^2 + 3y^2} \) so \( \langle f_x(1, 1), f_y(1, 1) \rangle = \langle 1, 3 \rangle / 2 \) and the equation of the tangent plane is \( z - 2 = (x - 1)/2 + 3(y - 1)/2 \).

(b) \( z = 0.1/2 + 3 \cdot 0.2/2 = 0.35 \) and the linear approximation is \( z = 2 + 0.35 = 2.35 \).

4. \( f_x = 3x^2 - 2x = 0 \) and \( f_y = 2y - 1 = 0 \) gives \( (x, y) = (0, 1/2) \) or \( (x, y) = (2/3, 1/2) \).

\( f_{xx} = 6x - 2, f_{yy} = 2 \) and \( f_{xy} = 0 \) so \( D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 4(3x - 1) \).

If \( (x, y) = (0, 1/2) \) then \( D = -4 < 0 \) so it is a saddle point.

If \( (x, y) = (2/3, 1/2) \) then \( D = 4 > 0 \) and \( f_{xx} = 2 > 0 \) so it is a local min.

5. Let \( F(x, y, z) = d(x, y, z)^2 = x^2 + y^2 + (z + 2)^2 \) be the square of the distance from \( (0, 0, -2) \) to \( (x, y, z) \) and let \( G(x, y, z) = z^2 - x^2 - y^2 - 1 \).

We want to maximize \( F(x, y, z) \) subject to the constraint \( G(x, y, z) = 0 \) and \( z > 0 \).

\( \nabla F(x, y, z) = \langle 2x, 2y, 2(z + 2) \rangle \) and \( \nabla G(x, y, z) = \langle -2x, -2y, 2z \rangle \) so we must find all solutions to

\[
2x = -\lambda 2x, \quad 2y = -\lambda 2y, \quad 2(z + 2) = \lambda 2z, \quad \text{and} \quad z^2 - x^2 - y^2 - 1 = 0
\]

We see that if \( \lambda \neq -1 \) then the first two equations gives \( x = y = 0 \) and the third equation cannot be solved if \( \lambda = 1 \): (i) If \( \lambda = 1 \) no solutions. (ii) \( \lambda = -1 \) then the first and second equation hold and the third equation gives \( z = -1 \) so we get \( (x, y, -1) \) and \( G(x, y, -1) = -x^2 - y^2 = 0 \) gives \( x = y = 0 \) so the point is \( (0, 0, -1) \) and \( F(0, 0, -1) = 1 \). If \( \lambda \neq -1 \) and \( \lambda \neq 1 \) then \( (x, y, z) = (0, 0, 1/(1 + \lambda)) \) and \( G(0, 0, 1/(1 + \lambda)) = 1/(1 + \lambda)^2 - 1 = 0 \) gives \( \lambda = 0 \) so the point is \( (0, 0, 1) \) and \( F(0, 0, 1) = 9 \). In conclusion the closest point is \( (0, 0, 1) \).