Lecture 20 (May 18th)

Today's Lecture: Non-Hom Systems of D.E.s.

$$\boxed{\underline{x}' = A\underline{x} + \underline{f}}$$ "Normal Form"

If $\underline{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} = \underline{0}$ the system is called "homogeneous"

If $\underline{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \neq \underline{0}$ the system is called "non-homogeneous"

To solve these types of systems we will use one of two methods:

1. Method of undet coeff

2. Variation of parameters

Solutions to non-hom DEs :

no $+\underline{f}$

Recall : Solutions to $\underline{x}' = A\underline{x}$ $\boxed{}$

$$\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t) + \cdots + c_n \underline{x}_n(t)$$

arbitrary constants (can be determined using initial conditions)

Different solutions to $\underline{x}' = A\underline{x}$

Constructed using e-values and e-vectors

But in the case of a non-hom system we need to add on a particular solution $\underline{x}_p(t)$

The goal of the method of undetermined coeff is to find this particular solution

General outline

1. Examine $f(t)$ to determine a guess for the particular solution
2. Substitute guess into equation
3. Use the resulting equation to determine any of the unknown coefficients

Examples

$$x_1'(t) = 2x_1(t) - t$$
$$x_2'(t) = -x_1(t) - x_2(t) + e^t$$

$$\Downarrow$$

$$\underline{x}' = A\underline{x} + \underline{f}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -t \\ e^t \end{pmatrix}$$

Depending on $f(t)$ we make a guess for $\underline{x}_p(t)$

① $\underline{f}(t) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ both numbers

$\Rightarrow$ Guess : $\underline{x}_p(t) = \underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

constant vector.

② $\underline{f}(t) = \begin{pmatrix} 2 \\ t^{1} \end{pmatrix}$

$\Rightarrow$ Guess :

$$\underline{x}_p(t) = \underline{a} + \underline{b}\,t^{1}$$

constant

vectors

③ $\underline{f}(t) = \begin{pmatrix} 2t^2 \\ t \end{pmatrix}$

$\Rightarrow$ Guess :

$$\underline{x}_p(t) = \underline{a} + \underline{b}\,t + \underline{c}\,t^2$$

constant vector.

exponential

④ $\underline{f}(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$

linear poly.

$$\underline{x}_p(t) = \underbrace{\underline{a}e^t}_{} + \underbrace{\underline{b} + \underline{c}t}_{}$$

deals w/ the exp

These are the "unknown coefficients"

deals w/ linear term

⑤ $\underline{f}(t) = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$ Two diff exponentials

$\Rightarrow$ Guess : $\underline{x}_p(t) = \underbrace{\underline{a}e^t}_{} + \underbrace{\underline{b}e^{-t}}_{}$

⑤* $\underline{f}(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$ Two related exponentials of exp... (same coeff in front of t)

$\Rightarrow$ Guess:

$$\underline{x}_p(t) = \underbrace{\underline{a} e^t}$$

deals with both components.

⑥ $\underline{f}(t) = \begin{pmatrix} e^{2t} \\ \cos(t) \end{pmatrix}$

$\Rightarrow$ Guess $\underline{x}_p = \underline{a} e^{2t} + \underline{b}\cos(t)$

$+ \underline{c}\sin(t)$

⑦ $\underline{f}(t) = \begin{pmatrix} e^{-t} \\ \sin(t) \\ \cos(2t) \end{pmatrix}$

Guess:

$$\underline{x}_p(t) = \underbrace{\underline{a}e^{-t}} + \underbrace{\underline{b}\sin(t) + \underline{c}\cos(t)}$$

$$+ \underbrace{\underline{d}\cos(2t) + \underline{e}\sin(2t)}$$

Example

Solve the DE:

$$\underline{x}' = \underbrace{\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}}_{A} \underline{x} + \begin{pmatrix} -9t \\ 0 \\ -18t \end{pmatrix}$$

1st Find the general sol$^{\underline{n}}$

to $$\underline{x}' = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \underline{x}$$

E-values: $\lambda = 3, 3, -3$

E-vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

3 Hom Solutions:

$$\underline{x}_1(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t}$$

$$\underline{x}_2(t) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{3t}$$

$$\underline{x}_3(t) = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} e^{-3t}$$

$\Rightarrow$ General Hom Sol$\underline{^{n}}$

is:

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{3t}$$

$$+ c_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} e^{-3t}$$

2nd

To find the particular solution: Use method of undet. coeff

$$\underline{f}(t) = \begin{pmatrix} -9t \\ 0 \\ -18t \end{pmatrix}$$

Guess:

$$\underline{x}_p(t) = \underline{a} + \underline{b}\,t$$

Now sub into the DE ($\underline{x} = \underline{x}_p$)

(will need to diff $\underline{x}_p$ before we can sub.)

$$\underline{x}_p'(t) = \frac{d}{dt}\left[\underline{a} + \underline{b}t\right]$$

$$= \underline{0} + \underline{b}$$

$$= \underline{b}$$

$$\boxed{\underline{x}_p'(t) = \underline{b}}$$

sub into $A\underline{x}$:

$$\underbrace{\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}}_{A} \underline{x} - p$$

$$= A(\underline{a} + \underline{b}t)$$

$$= \boxed{A\underline{a}} + \boxed{A\underline{b}}\,t$$

vector
(unknown
currently)

original DE :

$$\underline{x}' = A\underline{x} + \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix} t$$

Having substituted $\underline{x} = \underline{x}_p$ we got :

$$\underline{b} = \boxed{A\underline{a}} + \boxed{A\underline{b}}\,t + \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix} t$$

$$(+\underline{0}t)$$

compare coefficients on both sides of eq^{n}:

[1] constants $\quad \underline{b} = A\,\underline{a}$

[t] coeff of t $\quad \underline{0} = A\,\underline{b} + \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}$

1^{st} solve $\underline{0} = A\underline{b} + \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}$ for $\underline{b}$ (Then we'll solve for $\underline{a}$)

$$\underline{0} = A\underline{b} + \begin{pmatrix} -9 \\ 0 \\ -18 \end{pmatrix}$$

Rearrange

$$\begin{pmatrix} 9 \\ 0 \\ 18 \end{pmatrix} = A\underline{b}$$

i.e) $$\underbrace{\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 18 \end{pmatrix}}$$

2 perspectives

Matrix Eq^{n}

System of simultaneous Eq^{n}s

Solving we get :

$$\underline{b} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$$

see original lecture notes for details of calc.

Then we solve the 1^{st} Eq$^{\underline{n}}$:

$$A\underline{a} = \underline{b}$$

i.e. $\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$

Solving this we get:

$$\underline{a} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

see original lecture notes fw details of calc.

That means that:

$$\underline{x}_p = \underline{a} + \underline{b}\,t$$

$$= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} t$$

$$= \begin{pmatrix} 1 + 5t \\ 2t \\ 2 + 4t \end{pmatrix}$$

General Non-Hom Sol$^{\underline{n}}$:

$$\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{3t}$$

$$+ C_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1+5t \\ 2t \\ 2+4t \end{pmatrix}$$

General Homogeneous solution.

particular sol^n we found