# Lecture 21 (May 20th)

**Today's Lecture**: Non-Hom Systems of D.E.s.

$$\boxed{\underline{x}' = A\underline{x} + \underline{f}}$$ "Normal Form"

If $\underline{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} = \underline{0}$ the system is called "homogeneous"

If $\underline{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \neq \underline{0}$ the system is called "non-homogeneous"

To solve these types of systems we will use one of two methods:

1. Method of undet coeff (prev lecture)

2. Variation of parameters

**Recall (from 2nd order DEs)**

Idea: The particular solution $y_p(t)$ should look like:

$$y_p(t) = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)$$

Coeff functions

Diff Hom. sol$\underline{^n}$s

Method let's you calculate $v_1$ and $v_2$ using integration...

To replicate this technique in the context of systems of DEs:

$$\underline{x}_1(t), \underline{x}_2(t), \ldots, \underline{x}_n(t)$$

$$\boxed{\text{Sol}\underline{^n}\text{s to: } \underline{x}' = A\underline{x}}$$

The particular sol$\underline{^n}$:

$$\underline{x}_p(t) = v_1(t)\,\underline{x}_1(t) + v_2(t)\,\underline{x}_2(t) + \cdots + v_n(t)\,\underline{x}_n(t)$$

$$\underline{x}_p(t) = \underbrace{\begin{bmatrix} \underline{x}_1(t) & \underline{x}_2(t) & \cdots & \underline{x}_n(t) \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{bmatrix}}_{\underline{v}}$$

## Derivation of formula:

Since we assume $\underline{x}_p(t)$ is a particular sol$^{\underline{n}}$:

It satisfies $\underline{x}' = A\underline{x} + \underline{f}$

Differentiating $\underline{x}_p$:

$$X'(t)\,\underline{v}(t) + X(t)\,\underline{v}'(t)$$

Substituting $\underline{x}_p$ in RHS:

$$A\underline{x}_p + \underline{f} = A(X\underline{v}) + \underline{f}$$

Equate LHS and RHS:

$$\underbrace{\cancel{X'(t)\,\underline{v}(t)} + X(t)\,\underline{v}'(t)}_{\underline{x}_p'} = \underbrace{\cancel{A(X\underline{v})} + \underline{f}}_{A\underline{x}_p + \underline{f}}$$

Using the fact: $\boxed{X' = AX}$

$X$ is a fundamental matrix

Matrix ...

$$\Rightarrow \quad X(t)\,\underline{v}'(t) = \underline{f}(t)$$

$$\Rightarrow \quad \underline{v}(t) = \int X^{-1}(t)\,\underline{f}(t)$$

using fact $X$ is invertible

(since $\det(X) \neq 0$ $X$ is invertible)

Having calculated $\underline{v}(t)$:

$$\underline{x}_p(t) = \underbrace{X(t)}_{\text{collection of all sol}^n\text{s}}\ \underbrace{\underline{v}(t)}_{\text{calculated above.}}$$

hom

NON-HOM. SYSTEM

## Example

Solve the IVP:

$$\underline{x}' = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix} \underline{x} + \underbrace{\begin{pmatrix} e^{2t} \\ 1 \end{pmatrix}}_{\underline{f}}$$

$$\underbrace{\underline{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}}$$

will help us determine unknown coeff. $(c_1, c_2)$

General Solution to this DE:

$$\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t) + \underline{x}_p(t)$$

general hom. solution

particular solution

(1) Calculate e-values of

$$A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$$

$$\det \begin{pmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \lambda = -1, +1$$

(2) Calculate e-vectors assoc. to each e-value:

$\boxed{\lambda_1 = -1}$ * | $\boxed{\lambda_2 = 1}$ +

Solve:

$(A-(-1)I)\underline{u}=0$

$\begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}\underline{u}=\underline{0}$

Solving:

$\underline{u}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *

Solve:

$(A-(1)I)\underline{u}=0$

$\begin{pmatrix} 1 & -3 \\ 1 & -3 \end{pmatrix}\underline{u}=\underline{0}$

Solving:

$\underline{u}=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ *

Can now write down 2 sol$^{\underline{n}}$s to $\underline{x}'=A\underline{x}$ :

$$\underline{x}_1(t) = \begin{pmatrix}1\\1\end{pmatrix} e^{-1t}$$

$$\underline{x}_2(t) = \begin{pmatrix}3\\1\end{pmatrix} e^{1t}$$

$$\underbrace{X(t)}_{\uparrow} = \begin{bmatrix} \underline{x}_1(t) & \underline{x}_2(t) \end{bmatrix}$$

fundamental matrix

$$= \begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix}$$

$$\Rightarrow X^{-1}(t)$$

$$= \frac{1}{\det X}\begin{bmatrix} e^{t} & -3e^{t} \\ -e^{-t} & e^{-t} \end{bmatrix}$$

$$\det(X) = 1 - (3) = -2$$

$$\Rightarrow X^{-1}(t) = \begin{pmatrix} -\tfrac{1}{2}e^{t} & \tfrac{3}{2}e^{t} \\ \tfrac{1}{2}e^{-t} & -\tfrac{1}{2}e^{-t} \end{pmatrix}$$

3 Use formulae to calculate $\underline{x}_p(t)$

$$\underline{x}_p(t) = X(t)\,\underline{v}(t)$$

where $\underline{v} = \int X^{-1}(t)\,\underline{f}\,dt$

$$\int X^{-1}(t)\,\underline{f}\,dt$$

$$= \int \begin{pmatrix} -\frac{1}{2}e^{t} & \frac{3}{2}e^{t} \\ \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \end{pmatrix} \begin{pmatrix} e^{2t} \\ 1 \end{pmatrix} dt$$

$$= \int \begin{pmatrix} -\frac{1}{2}e^{3t} + \frac{3}{2}e^{t} \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \end{pmatrix} dt$$

$$\begin{pmatrix} -\frac{1}{2}e^{3t} + \frac{3}{2}e^{t} \end{pmatrix}$$

need

$$= \begin{pmatrix} \cdots \\ \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

No ... for $+C$

Calculate $\underline{x}_p$:

$$\underline{x}_p = X\,\underline{v}$$

$$= \begin{bmatrix} e^{-t} & 3e^{t} \\ e^{-t} & e^{t} \end{bmatrix} \begin{pmatrix} -\frac{1}{6}e^{3t} + \frac{3}{2}e^{t} \\ \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

$$= \begin{bmatrix} -\frac{1}{6}e^{2t} + \frac{3}{2} + \frac{3}{2}e^{2t} + \frac{3}{2} \\ -\frac{1}{6}e^{2t} + \frac{3}{2} + \frac{1}{2}e^{2t} + \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 3 + \frac{4}{3}e^{2t} \\ \cdots \end{bmatrix}$$

$$\left[\,2 + \tfrac{1}{3}e^{2t}\,\right] \;//$$

General (Non-Hom) Solution:

$$\underline{x}(t) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-1t}$$

$$+ C_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{1t}$$

$$+ \begin{bmatrix} 3 + \frac{4}{3}e^{2t} \\ 2 + \frac{1}{3}e^{2t} \end{bmatrix} \quad \underline{x}_p$$

To calculate $c_1$ and $c_2$ we apply:

$$\underline{x}(0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

sub $t=0$

$$\underline{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{0}$$

$$+ c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{0}$$

$$+ \begin{bmatrix} 3 + \frac{4}{3} e^{0} \\ 2 + \frac{1}{3} e^{0} \end{bmatrix}$$

$$\underline{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 + 4/3 \\ 2 + 1/3 \end{pmatrix}$$

Next rearrange:

$$\underline{x}(0) - \begin{pmatrix} 13/3 \\ 7/3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ \ldots \end{pmatrix} - \begin{pmatrix} 13/3 \\ \ldots \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \ldots \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ \ldots \end{pmatrix}$$

$$\begin{bmatrix} -16/3 \\ -7/3 \end{bmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Solving for $c_1, c_2$:

$$c_1 = -3/2$$

$$c_2 = -5/6$$

$$\underline{x}(t) = \left(-\frac{3}{2}\right)\begin{pmatrix}1\\1\end{pmatrix}e^{-1t} + \left(-\frac{5}{6}\right)\begin{pmatrix}3\\1\end{pmatrix}e^{1t} + \begin{bmatrix}3+\frac{4}{3}e^{2t}\\2+\frac{1}{3}e^{2t}\end{bmatrix}$$

$$= \begin{pmatrix}-3/2\,e^{-t}\\-3/2\,e^{-t}\end{pmatrix} + \begin{pmatrix}-\frac{5}{2}e^{t}\\-\frac{5}{6}e^{t}\end{pmatrix} + \begin{bmatrix}3+\frac{4}{3}e^{2t}\\2+\frac{1}{3}e^{2t}\end{bmatrix}$$

$$\underline{x}(t) = \underbrace{\begin{bmatrix} -\frac{3}{2}e^{-t} - \frac{5}{2}e^{t} + \frac{4}{3}e^{2t} + 3 \\ -\frac{3}{2}e^{-t} - \frac{5}{6}e^{t} + \frac{1}{3}e^{2t} + 2 \end{bmatrix}}_{\text{Solution to IVP}}$$

Solution to IVP