Lecture 8
Introduction to Differential Equations

Roman Kitsela

October 15, 2018
Chapter 2 (quick recap)

Chapter 4 - Linear second-order equations
  • Introduction (the mass-spring model)

Section 4.2 - Homogeneous linear equations
  • Introduction
  • Example
This lecture
• Auxiliary equation - reminder
This lecture

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- Example recap (two distinct roots)
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- A little theory (linear independence and the general solution)
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- The general solution for repeated root case and complex root case
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- Example
Quick Announcements + Questions

• First MATLAB assignment due Friday
• Homework 2 assigned on the website (also due Friday)
• Questions?
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\[ ay'' + by' + cy = 0 \]  \hspace{1cm} (1)
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\[ ar^2 + br + c = 0 \]  (2)
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**Shown last time:** 

\[ y(t) = e^{\lambda t} \]  is a solution to (1) **if and only if** \( r = \lambda \) is a solution to (2)
Example (from last lecture)

Example

Solve the differential equation

\[ y''(t) - 5y'(t) + 6y(t) = 0 \] (3)

Solution.

Characteristic equation:

\[ r^2 - 5r + 6 = 0 \]

\[ \Rightarrow r = 2 \text{ or } r = 3 \]

Two solutions to (3):

\[ y_1(t) = e^{2t} \text{ and } y_2(t) = e^{3t} \]

General solution:

\[ y(t) = C_1 e^{2t} + C_2 e^{3t} \]

Why exactly is this the general solution?
Example (from last lecture)

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**on** $(-\infty, \infty)$, **then there exists unique constants** $C_1$ and $C_2$ **such that**

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**satisfies the initial value problem**

$$ay'' + by' + cy = 0; \quad y(t_0) = Y_0, \quad y'(t_0) = Y_1$$

**on** $(-\infty, \infty)$
Theorem

If $y_1(t)$ and $y_2(t)$ are any two essentially different solutions to the D.E. then we can always find unique constants $C_1$ and $C_2$ such that $y(t) = C_1 y_1(t) + C_2 y_2(t)$ solves the original equation and satisfies any given initial conditions $y(t_0) = Y_0$, $y'(t_0) = Y_1$. 
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A pair of functions $y_1(t)$ and $y_2(t)$ is said to be **linearly independent on the interval $I$** if and only if neither of them is a constant multiple of the other on all of $I$. 

Quick Examples:

- $y_1(t) = e^{2t}$ and $y_2(t) = 4e^{2t}$ are linearly dependent (on $(-\infty, \infty)$).
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Definition (Linear Independence of functions)

A pair of functions $y_1(t)$ and $y_2(t)$ is said to be linearly independent on the interval $I$ if and only if neither of them is a constant multiple of the other on all of $I$. Otherwise we say that $y_1(t)$ and $y_2(t)$ are linearly dependent.
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Upshot of the theory

To solve second-order differential equations we need to find two "different" solutions $y_1$ and $y_2$ (different = linearly independent).

The general solution to a differential equation (without initial conditions) will then be given by a linear combination of $y_1$ and $y_2$:

$$y = C_1 y_1 + C_2 y_2$$

If we are given initial conditions (we need 2 for second-order problems!) we can always calculate $C_1$ and $C_2$ so that $y = C_1 y_1 + C_2 y_2$ will match those initial conditions.
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3. If we are given initial conditions (we need 2 for second-order problems!) we can always calculate $C_1$ and $C_2$ so that $y = C_1 y_1 + C_2 y_2$ will match those initial conditions.
We have seen that calculating solutions to $ay'' + by' + cy = 0$ amounts to solving the quadratic equation $ar^2 + br + c = 0$. There are three possible cases:

1. **Case 1** Two distinct roots $r_1$, $r_2$
2. **Case 2** One repeated root $r$
3. **Case 3** Two complex roots $\alpha + i\beta$, $\alpha - i\beta$

We have already seen that in **Case 1** $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are linearly independent solutions, and so the general solution has the form:

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Back to the auxiliary equation

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Blackboard plan

1. Derive a general solution in Case 2
2. Derive a general solution in Case 3

Examples
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Derive a general solution in Case 3

Examples