Math 20D - Fall 2008 - Final Exam

Name: ____________________________________

Student ID: ________________________________

Section time: ______________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books, calculators or telephones. You may use a "cheat sheet" of notes which should be at most one page.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 10 questions which are worth 200 points. You have 3 hours to complete the test.

Question	Score	Maximum
1		15
2		15
3		15
4		22
5		15
6		20
7		18
8		25
9		25
10		30
Total		200

Problem 1. [*15 points; 12, 3.*]

A colony $y(t)$ of yeast is growing in a bakery according to the differential equation

$$\frac{dy}{dt} = y^2(y^2 - 9), \;\; y(0) = y_0 > 0.$$

(i) Find the critical solutions, indicate their type, draw the phase line and sketch the graphs of some solutions.

(ii) For what initial values $y_0 > 0$ will the yeast colony eventually die out?

Problem 2. [*15 points.*]

Solve the initial value problem: $y' = 1 + 2xy,\ \ y(0) = 1$. (Your answer will require a definite integral.)

Problem 3. [*15 points.*]

Using undetermined coefficients, find the general solution of the differential equation

$$y'' - 2y' + 2y = 5\sin t.$$

Problem 4. [*22 points; 8, 4, 10.*]

Consider the differential equation

$$t^2 y'' - 3ty' + 3y = 0, \text{ for } t > 0.$$

(i) Find the values of r such that $y = t^r$ is a solution to the differential equation.

(ii) Check that $y_1 = t$ and $y_2 = t^3$ form a fundamental pair of solutions.

(iii) Find the general solution of the differential equation

$$t^2 y'' - 3ty' + 3y = t^3 \ln t.$$

Problem 5. [*15 points.*]

Using the Laplace transform, solve the initial value problem

$$y'' - 2y' + y = t^{10}e^t, \;\; y(0) = 1, y'(0) = 1.$$

Problem 6. [*20 points; 4, 8, 8.*]

The general solution of a certain first order system of differential equations $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{at} \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

where a is a *non-zero* real number.

(i) For what values of a is the origin a (proper) node? Will it be a source or a sink?

(ii) For what values of a is the origin a saddle equilibrium point? Carefully, draw the trajectories in this case.

(iii) For $a = 2$, find the matrix exponential e^{At}.

Problem 7. [*18 points.*]

Find the general real-valued solution of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \mathbf{x}.$$

Problem 8. [*25 points; 15, 6, 4.*]

Consider the differential equation

$$y'' + 2xy' + 2y = 0$$

whose solutions are power series in x centered at $x_0 = 0$.

(i) Find the recurrence relation between the coefficients of the power series y.

(ii) Write down the first three *non-zero* terms in each of the two linearly independent solutions.

(iii) What is the radius of convergence of the solutions which contains only even powers of x?

Problem 9. [*25 points; 15, 10.*]

(i) Find the inverse Laplace transform of the function

$$\frac{1}{(s+1)(s^2+4s+5)}.$$

(ii) Using the Laplace transform, solve the initial value problem

$$y'' + 4y' + 5y = e^{-t} + e^{-t+\pi} u_\pi(t), \quad y(0) = 0, y'(0) = 0.$$

Problem 10. [*30 points; 8, 22.*]

Two tanks A and B initially contain 2 gallons of fresh water. Water containing 2 lb salt/gallon flows into tank A at a rate of 3 gallons/minute. At the same time, water is drained from tank B at a rate of 3 gallon/minute.

The two tanks are connected by two pipes which allow water to flow in only one direction. Specifically, the first pipe allows water to flow from tank A into tank B at a rate of 4 gallons/minute. The second pipe allows water to flow from tank B into tank A at a rate of 1 gallon/minute.

(i) Let $Q_1(t)$ and $Q_2(t)$ be the quantities of salt (measured in pounds) in tanks A and B at time t. Show that

$$\mathbf{Q}' = \begin{bmatrix} -2 & \frac{1}{2} \\ 2 & -2 \end{bmatrix} \mathbf{Q} + \begin{bmatrix} 6 \\ 0 \end{bmatrix}.$$

(ii) Solve the system of differential equations (i) and determine the quantities $Q_1(t)$ and $Q_2(t)$ of salt present in each tank at time t. (Do not forget to take into account the initial conditions.) How much salt will each tank contain as time $t \to \infty$?