Math 20D - Fall 2011 - Final Exam

Name: ______________________________

Student ID: ______________________________

Section time: ______________________________

Instructions:

Please print your name, student ID and section time.

During the test, you may not use books, calculators or telephones. You may use a "cheat sheet" of notes which should be at most one page.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

You have 3 hours to complete the test.

Question	Score	Maximum
1		8
2		17
3		20
4		8
5		12
6		10
7		12
8		13
Total		100

Problem 1. [*8 points; 5, 3.*]

Consider the autonomous equation

$$\frac{dy}{dt} = y(600 - y) - 50,000, \;\; y(0) = y_0 > 0.$$

(i) Find the critical solutions, indicate their type, draw the phase line and sketch the graphs of some solutions.

(ii) Assume that $y(0) = 200$. What is the behavior of the solution in the long run?

Problem 2. [*17 points; 2, 2, 2, 2, 3, 6.*]

Consider the inhomogeneous differential equation

$$(\star)\quad x^2y'' - xy' + y = x\ln x, \text{ for } x > 0.$$

This problem has three main parts (A), (B), (C), all independent of each other.

(A.) Check that $y_1 = x$ is a solution to the homogeneous differential equation.

We now proceed to find a second solution y_2 to the homogeneous equation.

(B.1) Show that for any fundamental pair of solutions (y_1, y_2) to the homogeneous equation we must have $W(y_1, y_2) = Cx$ for some constant $C \neq 0$.

(B.2) Set $y_1 = x$. Consider a second solution y_2 to the the homogeneous equation satisfying the initial values

$$y_2(1) = 0,\ y_2'(1) = 1.$$

Show that $W(y_1, y_2) = x$.

(B.3) Use part (B.2) to show that the solution y_2 must satisfy

$$xy_2' - y_2 = x.$$

(B.4) Use (B3) to find a second solution y_2.

(C) Using the solutions

$$y_1 = x \text{ and } y_2 = x \ln x$$

to the homogeneous equation, find the general solution to the inhomogeneous equation ($\star$) by variation of parameters.

Problem 3. *[20 points; 3, 4, 4, 2, 7.]*

Consider the system $\vec{x}' = A\vec{x}$ where

$$A = \begin{bmatrix} -2 & -8 \\ 1 & -8 \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -6$. (You do not need to check this fact.)

(i) Find a fundamental pair of solutions to the system.

(ii) Draw the trajectories of the general solution. What is the type of the phase portrait you obtained?

(iii) Calculate the matrix exponential e^{At}.

(iv) Solve the initial value problem $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(v) Use variation of parameters to find a particular solution the following inhomogeneous system

$$\vec{x}' = Ax + \begin{bmatrix} 12t \\ \\ 0 \end{bmatrix}.$$

Problem 4. [*8 points.*]

Find two independent real valued solutions of the system

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -5 & 3 \end{bmatrix} \vec{x}.$$

Problem 5. [*12 points; 6, 3, 3.*]

Consider the differential equation

$$y'' - xy' - y = 0$$

whose solutions are power series in x centered at $x_0 = 0$.

(i) Find the recurrence relation between the coefficients of the power series y.

(ii) Write down the first three *non-zero* terms in each of the two linearly independent solutions.

(iii) Express the solution involving only even powers of x in closed form. The final answer should be a familiar exponential. You may need to recall the series expansion

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots + \frac{y^n}{n!} + \ldots.$$

Problem 6. [*10 points; 4, 6.*]

Consider the function

$$h(t) = \begin{cases} 0 & t < 1 \\ t^2 & 1 \leq t < 2 \\ t^2 + t - 2 & t \geq 2. \end{cases}$$

(i) Express h in terms of unit step functions.

(ii) Find the Laplace transform of h. You may leave your answer as a sum of fractions.

Problem 7. [*12 points.*]

Use Laplace transforms to solve the initial value problem

$$y'' + 2y' + 5y = e^{-2t}, \ y(0) = 0, y'(0) = 1.$$

Problem 8. [*13 points; 7, 3, 3*]

Consider the forcing function

$$h(t) = u_\pi(t) - u_{4\pi}(t).$$

(i) Solve the following initial value problem using Laplace transform

$$y'' + y = h(t),\ \ y(0) = y'(0) = 0.$$

(ii) Write your solution $y(t)$ explicitly over each of the three intervals

$$0 \leq t < \pi, \quad \pi \leq t < 4\pi, \quad 4\pi \leq t < \infty.$$

(iii) Draw the graph of the solution you found in (i).