

Math 20D - Fall 2011 - Midterm I

Name: \_\_\_\_\_

Student ID: \_\_\_\_\_

Section time: \_\_\_\_\_

**Instructions:**

Please print your name, student ID and section time.

During the test, you may not use books, calculators or telephones. You may use a "cheat sheet" of notes which should be at most half a page, front and back.

Read each question carefully, and show all your work. Answers with no explanation will receive no credit, even if they are correct.

There are 6 questions which are worth 60 points. You have 50 minutes to complete the test.

| Question | Score | Maximum |
|----------|-------|---------|
| 1        |       | 10      |
| 2        |       | 10      |
| 3        |       | 10      |
| 4        |       | 10      |
| 5        |       | 10      |
| 6        |       | 10      |
| Total    |       | 60      |

**Problem 1.** [10 points; 4, 6.]

Consider the differential equation

$$(4x^3y^3 + ay^2) + (3x^4y^2 + 2xy + 2)y' = 0,$$

where  $a$  is a constant.

(i) Show that the differential equation is exact if and only if  $a = 1$ .

(ii) Solve the differential equation, when exact. It suffices to give the solution implicitly.

**Problem 2.** [10 points.]

A colony  $y(t)$  is growing in a bakery according to the differential equation

$$\frac{dy}{dt} = y^2 - 5y + 6.$$

Determine the critical points and indicate their type i.e. asymptotically stable, unstable, semistable. Sketch the phase line. What is the long-term behavior of the solution satisfying the initial value  $y(0) = 1$ ?

**Problem 3.** [10 points.]

Find the general solution of the first order equation

$$t^3 y' + 4t^2 y = 2 \sin t.$$

**Problem 4.** [10 points; 5, 5.]

- (i) Write down a second order constant coefficient homogeneous differential equation admitting the solution

$$y = e^{3t}(\cos t + 4 \sin t).$$

- (ii) For the same differential equation, solve the initial value problem

$$y(0) = 1, \quad y'(0) = 2.$$

You do not need to answer part (i) in order to solve part (ii).

**Problem 5.** [10 points; 7, 3.]

Consider the differential equation

$$t^2y'' - 3ty' + 3y = 0, \text{ for } t > 0.$$

(i) Check that  $y_1 = t$  and  $y_2 = t^3$  form a fundamental pair of solutions.

(ii) Write down the general solution to the differential equation.

**Problem 6.** [*10 points.*]

A swimming pool originally contains 100 gallons of fresh water. Water containing 5 lb of dye per gallon is poured into the pool at a rate of  $r$  gal/min. The mixture is allowed to leave the pool at the same rate. After 100 minutes, dye in the pool is measured, and found to be exactly 50 lb. What is the rate water was poured into the pool?