Math 20D **Name:** ____________________

Dec 13, 2018 **Final** **PID:** ____________________

Instructions

1. **Write your *Name* and *PID* on the front of your Blue Book.**
2. **No calculators or other electronic devices are allowed during this exam.**
3. **You may use a double sided page of notes.**
4. **Write your solutions clearly in your Blue Book.**
   (a) **Carefully indicate the number and letter of each question and question part.**
   (b) **Present your answers in the same order as they appear in the exam.**
   (c) **Start each numbered problem on a new side of a page.**
5. **Show all of your work and justify all your claims. No credit will be given for unsupported answers, even if correct.**

---

1. (10 points) Solve the following integral equation

$$y(t) + \int_0^t e^{t-\nu} y(\nu) d\nu = \sin(t)$$

2. (10 points) Determine the correct form for a particular solution to the following non-homogeneous differential equations
   (a) $y'' - 2y' + 2y = te^t \cos(t)$
   (b) $y'' = t^2 - t + 1$
   (c) $y'' - y = \cos(t) - \sin(t) + \sin(2t)$

   (**You do not need to calculate coefficients**)

3. (10 points) Solve the symbolic initial value problem

$$\frac{d^2x}{dt^2} + 4x = 6\delta(t-\pi); \quad x(0) = 2, \ \frac{dx}{dt}(0) = 0$$

4. (10 points) By finding the generalized eigenvalues or otherwise, find the general solution to the homogeneous system

$$\mathbf{x}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}$$

5. (10 points) Suppose y(t) is a solution to the initial value problem

$$ty'' + 2(t-1)y' - 2y = 0; \quad y(0) = 0, y'(0) = 0$$

   Find the Laplace transform of $y(t)$

6. (10 points) Consider the following discontinuous function

$$g(t) = \begin{cases} 0 & 0 \le t < 1 \\ t^2 & 1 \le t < 2 \\ t^2 + t - 2 & t \ge 2 \end{cases}$$

(a) Express $g(t)$ in terms of unit step functions $u(t)$

(b) Find the Laplace transform of $g(t)$

7. (10 points) Find a general solution to the following non-homogeneous system of differential equations

$$\frac{dz_1}{dt}(t) = 6z_1(t) + z_2(t) - 11$$
$$\frac{dz_2}{dt}(t) = 4z_1(t) + 3z_2(t) - 5$$

8. (10 points) Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}, \quad y(1) = 1$$

9. (10 points) (a) Verify that $\left\{ \begin{pmatrix} e^{7t} \\ 2e^{7t} \end{pmatrix}, \begin{pmatrix} e^{-5t} \\ -2e^{-5t} \end{pmatrix} \right\}$ is a fundamental solution set to the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 12 & 1 \end{pmatrix} \mathbf{x}$$

(b) Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 12 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

(c) Given a system of differential equations $\mathbf{x}' = A\mathbf{x}$ with fundamental matrix $X(t)$, the matrix exponential function e^{At} satisfies

$$e^{At} = X(t)X(0)^{-1}$$

where $X(0)^{-1}$ denotes the inverse matrix to $X(0)$. Using part (a) or otherwise, calculate e^{At}

10. (10 points) Find a particular solution to the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = \frac{e^t}{t}$$

11. (10 points) Find an explicit solution to the initial value problem

$$e^{2y}\frac{dy}{dx} = 8x^3, \quad y(1) = 0$$

For which values of x is this solution valid?